Math 685. Lecture 13

Proj 4 :

→ midpoint method in 2.1

$$y_{k+1} = y_k + h_k f\left(t_k + \frac{h_k}{2}, \frac{y_k + y_{k+1}}{2}\right)$$

→ in Prob. 2.2 : fix $x = \frac{L}{2}$.

BVPs. Shooting.

$$\begin{cases} x'' = f(t, x, x'), & a < t < b \\ x(a) = \alpha, \; x(b) = \boxed{\beta} \end{cases}$$

$$\begin{cases} x'' = f(t, x, x') \\ x(a) = \alpha, \; x'(a) = z \end{cases} \leftarrow \text{Solve this IVP}$$

$\varphi(z)$ s.t. $\varphi(b) = x(b)$

solution to IVP

Want: $\varphi(b) = \boxed{\beta}$

Ex. $\begin{cases} x'' = x \\ x(0) = 1 \quad x(1) = \underline{\underline{7}} \end{cases}$

1) IVP : $\begin{cases} x'' = x \\ x(0) = 1, \; x'(0) = z \end{cases}$

2) $\varphi(z)$ – solution to IVP with slope z.

$x'' = x$ ⊛

$x(t) = C_1 e^{t} + C_2 e^{-t}$ ← general sol. of ⊛

$\varphi(z)$ ← $x(t)$ subject to initial conditions.

$x(0) = 0, \quad x'(0) = z$

$$x(t)\Big|_{IVP} = \varphi(z) = \frac{1}{2}(1+z)e^{t} + \frac{1}{2}(1-z)e^{-t}$$

3) Solve for z from $\varphi(z)\Big|_{t=1} = 7$

$$\frac{1}{2}(1+z)e + \frac{1}{2}(1-z)e^{-1} = 7 \Rightarrow z \text{ value that makes } IVP \equiv BVP.$$

4) Plug this value into $\varphi(z)$ to get solution to BVP.

Finite diff. schemes

give discontinuous discrete solution.

$u(x_i, t_j) = u_{ij}$

Usually results in sparse system of linear equations.

Collocation methods.

solution = comb. of continuous functions

$$u(x,t) \approx v(\vec{x},t) = \sum_{i=1}^{n} x_i \varphi_i(t)$$

x_i – unknown coefficients

$\varphi_i(t)$ – basis functions.

Pick $a = t_1 < t_2 < \dots < t_n = b$

t_i – collocation points

At each t_i, ODE has to hold:

$$\begin{cases} v''(\vec{x}, t_i) = f(t, v, v') & \leftarrow \text{at interior pts} \\ v(\vec{x}, t_1) = \alpha \\ v(\vec{x}, t_n) = \beta \end{cases}$$

for system

$$\begin{cases} u'' = f(t, u, u') \\ u(a) = \alpha \\ u(b) = \beta \end{cases}$$

Ex. $\begin{cases} u'' = 6t, & u(0) = 0 \\ & u(1) = 1 \end{cases}$

t_1, t_2, t_3 – collocation pts: 0, $\frac{1}{2}$, 1

Pick basis: $v(\vec{x}, t) = x_1 + x_2 t^2 + x_3 t^3$

basis $\varphi_1 \equiv 1$, $\varphi_2 = t^2$

$v' = 2t x_2 + 3t^2 x_3$, $v'' = 2x_2 + 6t x_3$

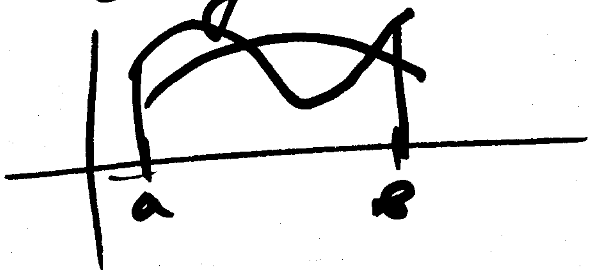
$$\begin{cases} v''(t_2) = 6t_2 = 2x_2 + 3x_3 \Rightarrow \cancel{x_2 = \frac{3}{2}} \\ v(t_1) = 0 = x_1 + x_2 \cdot 0 \Rightarrow x_1 = 0 \\ v(t_3) = 1 = x_1 + x_2 \cdot 1 + x_3 \Rightarrow x_2 + x_3 = 1 \end{cases}$$

$$\boxed{v = t^3} \Leftarrow \begin{cases} 3 = 2x_2 + 3x_3 \\ 1 = x_2 + x_3 \quad (\cdot 2) \end{cases} \Rightarrow 1 = x_3 \quad x_2 = 0.$$

Collocation methods

Spectral

φ_i are nonzero everywhere on $[a, b]$

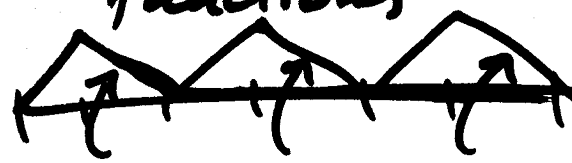


+ accurate results for reasonable # of grid pts
- ~~stiff~~ dense systems of linear equations, can be ill-conditioned.

  comp. complexity is high (can be alleviated by using orthogonal basis functions)

finite element methods

localized basis functions



φ_1 is zero where φ_2, φ_3 are nonzero.

- to achieve accuracy, need many points
+ sparse systems of equations well-conditioned tri-diagonal $O(n)$-complexity

Galerkin method.

Instead of imposing ODE condition at collocation pts (residual at t_i to be exactly $=0$), minimise residual everywhere.

$$u(x,t) \approx v(x,t) = \sum_i x_i \varphi_i(t)$$

$$r(t,x) = \underbrace{v''(x,t) - f(t,v,v')}_{\text{residual of ODE.}}$$

$$F = \frac{1}{2}\int_a^b r(t,x)^2\,dt \to \min.$$

$\Rightarrow Ax = b,\quad a_{ij} = \int \varphi_i'' \varphi_j'' dt$ - stiffness matrix

$b_i = \int f \varphi_i'' dt$ - load vector

$\downarrow$

symm. matrix

weighted residual method instead.

$\Rightarrow$ not symmetric system $Ax=b$.

Galerkin idea: pick $w_i = \varphi_i$

gain back symmetry, stiffness matrix has only 1st order derivatives:

$$a_{ij} = -\int \varphi_i' \varphi_j'$$

$$b_i = +\int f \varphi_i'$$

Implicit methods are preferred for stiff systems, can be unconditionally stable → work for large time step h.

$$\boxed{y_{k+1}} = y_k + h_k f\left(t_k + \frac{h_k}{2}, \frac{y_k + \boxed{y_{k+1}}}{2}\right)$$

nonlinear solver like Newton to solve for y_{k+1}.

Partial differential equations.

$f(t,x) = tx^2$ $\quad \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}$

partial