**Math 685/CSI 700/OR 682 Take-home Exam**
**given 03/04/08, due in class 03/24/08**

**Problem 1.**
Show how to evaluate the following expressions in a numerically stable fashion:

$$\text{(a)}\ \frac{1}{1+2x} - \frac{1-x}{1+x}, \qquad \text{for} \quad |x| \ll 1$$

$$\text{(b)}\ \sqrt{x+\frac{1}{x}} - \sqrt{x-\frac{1}{x}}, \qquad \text{for} \quad x \gg 1$$

**Problem 2.**
Prove that if Gaussian elimination with partial pivoting is applied to a matrix $A$ that is diagonally dominant by columns, then no row interchanges will occur.

**Problem 3.**

$$A = \begin{pmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{pmatrix} \text{ and } b = \begin{pmatrix} 0.217 \\ 0.254 \end{pmatrix}.$$

The exact solution of $Ax = b$ is $x = (1, -1)^T$. Further, let two approximate solutions $x_1 = (0.999, -1.001)^T$, $x_2 = (0.341, -0.087)^T$ be given.
(a) Does the more accurate solution have a smaller residual?
(b) Determine the exact $A^{-1}$ and $\mathtt{cond}(A)$ with respect to the maximum norm.
(c) Explain the discrepancy observed in (a) by linking the residuals to the corresponding backward error.

**Problem 4.**
Let $A^TAx = A^Tb$, $(A^TA+F)\tilde{x} = A^Tb$, $2||F||_2 \leq \sigma_n(A)^2$, where $\sigma_n(A)$ denotes the smallest singular value of $A$. Show that if $r = b - Ax, \tilde{r} = b - A\tilde{x}$, then $\tilde{r} - r = (I - A(A^TA+F)^{-1}A^T)Ax$ and consequently

$$||\tilde{r} - r||_2 \leq 2\mathtt{cond}(A)\frac{||F||_2}{||A||_2}||x||_2.$$

What does this estimate accomplish? Hint: Notice the role of $(A^TA)^{-1}F$ in the above estimate and use the relationship $||(A^TA)^{-1}||_2 \cdot ||A||_2^2 = \mathtt{cond}(A)^2$. The following fact from matrix algebra might be useful: for any $M$ s.t. $||M|| < 1$, $||(I+M)^{-1}|| \leq \dfrac{1}{1-||M||}$.

**Problem 5.**
Matrix $C = (A^TA)^{-1}$ with $\mathtt{rank}(A) = n$ is frequently used in statistics, where it is called a covariance matrix. Suppose we have obtained a QR decomposition $A = QR$.
(a) Prove that $C = (R^TR)^{-1}$.
(b) Design an algorithm for computing the diagonal entries of $C$ requiring $n^3/3$ or less floating point operations.
Hint: Use the special structure of $C$ and look for optimal algorithms in this context.

**Problem 6. Computer part.**
Consider a boundary value problem for the 2nd order differential equation:

$$u_{xx} - u = f(x), x \in [0,1]$$
$$u(0) = 0, u(1) = 0,$$

where $u = u(x)$ is the unknown and $f = f(x)$ is a given right-hand side. To solve this problem numerically, we first partition the interval $0 \leq x \leq 1$ into $N$ equal subintervals and thus build a uniform grid of $N+1$ nodes: $x_j = j \cdot h, h = 1/N, j = 0, 1, \dots N$. Then, instead of looking for the continuous function $u = u(x)$ we will be looking for its approximate table of values $u_0, \dots, u_N$ at the grid nodes $x_0, x_1, \dots, x_n$ respectively. At every interior node $x_j, j = 1, \dots, N-1$ we approximately replace the second derivative by the difference quotient

$$(u_{xx})|_{x=x_j} \approx \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2},$$

and arrive at the following finite-difference counterpart of the original problem (a central difference scheme):

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} - u_j = f_j, j = 1, 2, \dots, N-1$$
$$u_0 = 0, u_N = 0$$

where $f_j = f(x_j)$ is the discrete right-hand side assumed given, and $u_j$ is the unknown solution.

(a) Write down the previous scheme as a system of $(N-1) \times (N-1)$ linear algebraic equations in matrix form.
(b) Show that the system is tri-diagonal and at least non-strictly diagonally dominant, so a tri-diagonal elimination without pivoting can be applied.
(c) Implement the tri-diagonal elimination MATLAB. Solve the system for the right-hand side $f(x) = (-\pi^2 \sin(\pi x) + 2\pi \cos(\pi x))e^x$, for which the exact solution $u(x) = \sin(\pi x)e^x$ is known. For a sequence of grids with $N = 32, 64, 128, 256$ and $512$, compute relative error in maximum norm:

$$e(N) = \frac{\max_{1 \leq j \leq N-1} |u(x_j) - u_j|}{\max_{1 \leq j \leq N-1} |u(x_j)|}.$$

Plot $\log_2(e(N))$ vs. $\log_2 N$ and show that by reducing the grid size by a factor of 2 we decrease the error by a factor of 4 (second order convergence). Discuss accuracy and stability.

**Problem 7. Extra credit.**
Let $A$ be an $n \times n$ nonsingular matrix with QR decomposition $A = QR$. Determine the conditions under which the ratio $||y||_2/||x||_2$ with $x, y$ satisfying relationships $R^T x = d, Ry = x$ approximates $||A^{-1}||_2$. Hint: look at the SVD decomposition for $A$, i.e. $A = USV^T$, and expand vectors $d, x$ and $y$ on the basis spanned by suitable vectors $u_i$ or $v_i$.