Math 678.
Lecture 9

IVP: $\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy \quad \circledast$$

Thm: $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \Rightarrow \circledast$ satisfies:

1) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$
2) $u_t = \Delta u$, $t>0$
3) $\displaystyle\lim_{\substack{x\to x^0,\, t\to 0 \\ t>0}} u(x,t) = g(x^0)$, $x^0 \in \mathbb{R}^n$

Nonhomogeneous heat eqn:

$$\begin{cases} u_t - \Delta u = f, & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

Duhamel's principle

1) **In linear algebra:** let $u_0, u_1, \ldots$ in $\mathbb{R}^n$ satisfy $\boxed{u_{j+1} = A_{j+1} u_j}$

$A_1, A_2, \ldots$ - sequence of operators on $\mathbb{R}^n$.

$$u_k = A_k A_{k-1} \cdots A_1 u_0$$

$$u_k = \underbrace{A_k A_{k-1} \cdots A_{j+1}}_{S(k,j)} u_j$$

if $k > j$ $\qquad S(1,1) = I$

$$\Rightarrow \boxed{u_k = S(k,j) u_j}$$

Nonhomogeneous problem: $\boxed{u_{j+1} = A_{j+1} u_j + v_{j+1}}$

$v_1, v_2, \ldots$ - sequence of given vectors

$$u_1 = A_1 u_0 + v_1 = S(1,0)u_0 + v_1$$

$$u_2 = A_2 u_1 + v_2 = A_2(A_1 u_0 + v_1) + v_2 =$$
$$= A_2 A_1 u_0 + A_2 v_1 + v_2$$
$$= S(2,0)u_0 + S(2,1)v_1 + v_2$$

$$u_3 = A_3 u_2 + v_3 = A_3(A_2 A_1 u_0 + A_2 v_1 + v_2) + v_3$$
$$= A_3 A_2 A_1 u_0 + A_3 A_2 v_1 + A_3 v_2 + v_3$$
$$= S(3,0)u_0 + S(3,1)v_1 + S(3,2)v_2 + v_3$$

$$u_k = S(k,0)u_0 + S(k,1)v_1 + S(k,2)v_2 + \dots \Rightarrow$$

$$\boxed{u_k = S(k,0)u_0 + \sum_{j=1}^{k} S(k,j)v_j}$$

2) <u>In ODE</u>: $\frac{du}{dt} = A(t)u(t)$, $t > 0$ (1)

if $u(t) = S(t,t_0)u(t_0)$ solves (1) with $u(0) = g$.
and $A(t)$ - continuous map

Then solution to $\begin{cases} \frac{du}{dt} = A(t)u(t) + f(t), & t > 0 \\ u(0) = g \end{cases}$

is given by $u(t) = S(t,0)u(0) + \int_0^t S(t,s)f(s)\,ds$

<u>Justification</u>: $\frac{d}{dt}(S(t,t_0)u(t_0)) = A(t)\cdot S(t,t_0)u(t_0)$

~~$\frac{du}{dt} =$~~ $\frac{d}{dt}(S(t,0)u(0)) = A(t)\cdot S(t,0)u(0)$

$$\Rightarrow \frac{du}{dt} = A(t)S(t,0)u(0) + \int_0^t A(t)S(t,s)f(s)\,ds + \underbrace{S(t,t)}_{= I}\underbrace{f(t)}_{\int_0^t \frac{df}{ds}ds}$$

$$\Rightarrow \frac{du}{dt} = A(t)\cdot\left[S(t,0)u(0) + \int_0^t S(t,s)f(s)\right] + f(t)$$

$$\frac{du}{dt} = A(t)u(t) + f(t)$$

3) <u>In PDE case:</u>

$$u = u(x,t;s) = \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\,dy$$

solves $\begin{cases} u_t(\cdot;s) - \Delta u(\cdot;s) = 0, & \text{in } \mathbb{R}^n \times (s,\infty) \\ u(\cdot;s) = f(\cdot;s) & \text{on } \mathbb{R}^n \times \{t=s\} \end{cases}$

IVP with time starting at $t=s$

Consider the non-homogeneous problem:

$$\begin{cases} u_t - \Delta u = f, & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases} \quad (**)$$

By Duhamel's principle we obtain $u(x,t)$ - solution to $(**)$ - by integrating $u(x,t;s)$:

$$u(x,t) = \int_0^t u(x,t;s)\,ds, \qquad x \in \mathbb{R}^n, t > 0$$

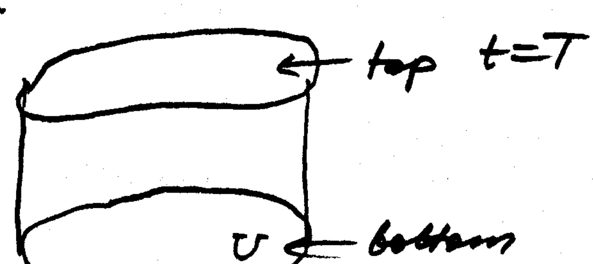
$$= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\,dy\,ds =$$

$$= \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \cdot \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\,dy\,ds$$

Consequence: $\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$

$$\Rightarrow u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\,dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\,dy\,ds$$

<u>Mean Value formula</u>.



$U_T := U \times (0,T]$ - parabolic cylinder

$\Gamma_T := \overline{U}_T - U_T$ - parabolic boundary = bottom + sides

U_T – parabolic interior – includes the top.

$\Phi(x-y)$ – fund. sol. of Laplace is constant on any $\underline{\text{sphere } \partial B(x,r)}$
level set of Φ

$$E(x,t;r) := \{(y,s) \in \mathbb{R}^{n+1} \mid s \le t,\ \Phi(x-y, t-s) \ge \tfrac{1}{r^n}\}$$

"heat ball"

fund. sol. of heat eqn

∂E is the level set for Φ

$\underline{\text{Thm}}$: (MVT for heat eqn)

$u \in C_1^2(U_T)$ – the soln to heat eqn

Then $$u(x,t) = \frac{1}{4r^n} \iint_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}\, dy\, ds \quad \forall E(x,t;r) \subset U_T$$

$\underline{\text{Proof}}$: Change coordinates to $x=0$, $t=0$.

$\Rightarrow E(r) = E(0,0;r)$

Assume u is smooth (otherwise use a mollifier)

$$\varphi(r) := \frac{1}{r^n} \int_{E(r)} u(y,s) \frac{|y|^2}{s^2}\, dy\, ds = \iint_{E(1)} u(ry, r^2 s) \frac{|y|^2}{s^2}\, dy\, ds$$

$$\varphi'(r) = \iint_{E(1)} \left(\sum_{i=1}^{n} u_{y_i} \cdot y_i \cdot \frac{|y|^2}{s^2} + 2r \cdot u_s \cdot \frac{|y|^2}{s^2} \right) dy\, ds$$

$$= \underbrace{\frac{1}{r^{n+1}} \iint_{E(r)} \left(\sum u_{y_i} \cdot y_i \frac{|y|^2}{s^2}\right.}_{I} \underbrace{\left. + 2u_s \frac{|y|^2}{s^2} \right) dy\, ds}_{J}$$

$$\psi := \frac{-n}{2} \log(-4\pi s) + \frac{|y|^2}{4s} + n \log r$$

(1) $\psi = 0$ on $\partial E(r)$ since ∂E has $\Phi(-y,-s) = \frac{1}{r^n}$

on $\partial E(r)$: $\Phi(-y,-s) =$

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & t>0 \\ 0, & t<0 \end{cases}$$

$$= \frac{1}{(-4\pi s)^{n/2}} e^{+|y|^2/4s} = \frac{1}{r^n}$$

$$e^{|y|^2/4s} = \frac{(-4\pi s)^{n/2}}{r^n}$$

$$\frac{|y|^2}{4s} = \frac{n}{2}\log(-4\pi s) - n\log r \quad \text{on } \partial E(r)$$

$$\Rightarrow \psi = -\frac{n}{2}\log(-4\pi s) + \frac{|y|^2}{4s} + n\log r = 0 \quad \text{on } \partial E(r).$$

(2) $\frac{\partial \psi}{\partial y_i} = \frac{y_i}{2s} \Rightarrow$

Now $$I = \frac{1}{r^{n+1}} \iint_{E(r)} \sum u_{y_i} \cdot y_i \cdot \frac{|y|^2}{s^2}\, dy\, ds =$$

$$= \underline{\frac{1}{r^{n+1}} \iint 4u_s \sum u_{y_i} \cdot \psi_{y_i}\, dy\, ds}$$