Math 678.
Lecture 5

Liouville Thm:

$u: \mathbb{R}^n \to \mathbb{R}$ harmonic, bounded $\Rightarrow u \equiv \text{const}$.

Proof:

$$u(x_0) = \fint_{B(x_0,r)} u(y)\,dy \quad \forall B(x_0,r)$$

u – harmonic

$u \in C^2(U) \cap C(\bar{U})$, harmonic $\Rightarrow u \in C^\infty$

follows from $u: \mathbb{R}^n \to \mathbb{R}$ bounded

$$\Delta u = 0 \Rightarrow \Delta u_{x_i} = 0 \quad i = 1, \dots n$$

u_{x_i} – also satisfies MVT:

$$u_{x_i}(x_0) = \fint_{B(x_0,r)} u_{x_i}(y)\,dy =$$

$$= \frac{1}{\alpha(n) r^n} \int_{B(x_0,r)} u_{x_i}(y)\,dy \underset{\text{Div Thm}}{=} \frac{1}{\alpha(n) r^n} \int_{\partial B(x_0,r)} u \nu_i \, dS(y)$$

$$|u_{x_i}(x_0)| = \frac{1}{\alpha(n) r^n} \left| \int_{\partial B(x_0,r)} u \nu_i \, dS(y) \right| \leq$$

$$\leq \frac{1}{\cancel{\alpha(n)} \cdot r^n} \|u\|_{L^\infty(\partial B(x_0,r))} \cdot \underset{= 1}{|\nu_i|_{L^\infty}} \underbrace{n \cancel{\alpha(n)} r^{n-1}}_{\text{surface area}}$$

$$\leq \frac{n}{r} \|u\|_{L^\infty(\mathbb{R}^n)} \leq \frac{Cn}{r} \text{ for some } C > 0.$$

$$\lim_{r \to +\infty} |u_{x_i}(x_0)| \leq \lim_{r \to +\infty} \frac{Cn}{r} = 0 \Rightarrow u_{x_i}(x_0) = 0$$

$\forall x_0 \in \mathbb{R}^n$, $i = 1, \dots n \Rightarrow u \equiv \text{const}$.

Representation formula (for $n \geq 3$)

If $f \in C_c^2(\mathbb{R}^n)$, $n \geq 3$
compact support

Any bounded solution of $-\Delta u = f$ in $\mathbb{R}^n$ has the form $u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy + C$ for some C.

Proof:

We can show $\Phi(x) \to 0$ as $|x| \to \infty$, $n \geq 3$

$$\Phi(x) = \frac{K}{|x|^{n-2}}, \quad K = \frac{1}{n(n-2)\alpha(n)}$$

To show: $\bar{u}(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$ is bounded.

$$|\bar{u}(x)| = \left| \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy \right| =$$

$$= K \left| \int_{\mathbb{R}^n} \frac{f(y)\, dy}{|x-y|^{n-2}} \right| = K \left| \int_{B(x,\varepsilon)} \frac{f(y)\, dy}{|x-y|^{n-2}} + \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{f(y)\, dy}{|x-y|^{n-2}} \right|$$

$$\leq K \|f\|_{L^\infty(B(x,\varepsilon))} \left| \int_{B(x,\varepsilon)} \frac{dy}{|x-y|^{n-2}} \right| + C \cdot K \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \underbrace{|f(y)|\, dy}_{\text{finite, compact supp}}$$

$$\int_{B(x,\varepsilon)} \frac{dy}{|x-y|^{n-2}} = \int_0^\varepsilon \int_{\partial B(x,r)} \frac{dS}{|x-y|^{n-2}}\, dr = n\alpha(n) \int_0^\varepsilon r\, dr < \infty$$

$$\int_0^\varepsilon \frac{1}{r^{n-2}} \cdot n \cdot r^{n-1} \alpha(n)\, dr$$

So $\bar{u}(x)$ is bounded.

Any other bounded solution of $-\Delta u = f$ will give $w(x) := \bar{u}(x) - u(x)$ as another bounded function satisfying $\Delta w = 0$ in $\mathbb{R}^n \Rightarrow w \equiv \text{const}$.

Analyticity:

Thm: u- harmonic in $U \Rightarrow u$ - analytic in U.

Harnack's inequality:

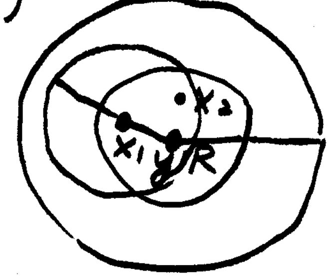
If V - connected open set in U s.t.

$$V \subset \underset{\text{compact}}{\overline{V}} \subset U$$

then $\exists C = C(V)$ s.t. $\sup\limits_V u \le C \inf\limits_V u$

for all non-negative harmonic functions in U.

Proof: Take $\forall x_1, x_2 \in B_R(y)$



$$B_R(x_1) \subset B_{2R}(y) \subset B_{3R}(x_2)$$

$$u(x_1) = \frac{1}{\alpha_n \cdot R^n} \int_{B_R(x_1)} u(x)\, dx \le$$

$$\le \frac{1}{\alpha_n \cdot R^n} \cdot \int_{B_{2R}(y)} u(x)\, dx \le \frac{1}{\alpha_n \cdot R^n} \cdot \int_{B_{3R}(x_2)} u(x)\, dx =$$

$$= \frac{1}{\cancel{\alpha_n} R^n} \cdot u(x_2) \cdot \cancel{\alpha_n} \cdot (3R)^n = 3^n \cdot u(x_2)$$

Since x_1, x_2 - arbitrary $\Rightarrow$ $u(x_1) \le \underset{\overset{\parallel}{3^n}}{C} \cdot u(x_2)$

$\Rightarrow \sup\limits_V u \le C \cdot \inf\limits_V u$

Corollary: $\forall x, y \in V$ $\quad \frac{1}{C} u(y) \le u(x) \le C u(y)$

Values of u have to be comparable on the entire domain U where u is harmonic and non-negative, away from ∂U.

Local estimates of gradients of harmonic fcts:

<u>Thm</u>: $\Delta u = 0$ in $U \Rightarrow$

$$|D^\alpha u(x_0)| \le \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))} \qquad \forall B(x_0,r) \subset U$$
$$\forall \alpha \text{ with } |\alpha| = k$$

$$C_k = \begin{cases} \dfrac{1}{\alpha(n)}, & k = 0 \\[2ex] \dfrac{(2^{n+1} n k)^k}{\alpha(n)}, & k = 1, \ldots \end{cases}$$

<u>Proof</u>:

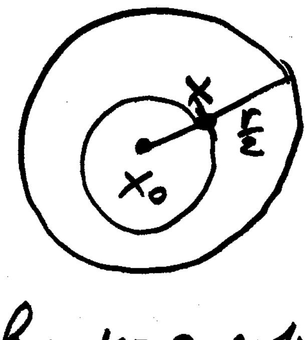
$\boxed{k=0}$: $$|D^0 u(x_0)| = |u(x_0)| = \Big|\fint_{B(x_0,r)} u\, dy\Big| \le \frac{1}{\alpha(n)\cdot r^n} \|u\|_{L^1(\underset{\text{Ball}}{B})}$$

$\boxed{k=1}$: $\Delta u_{x_i} = 0$

$$|u_{x_i}(x_0)| \underset{\text{MVT}}{=} \Big|\fint_{B(x_0,\frac{r}{2})} u_{x_i}\, dx\Big| = \Big|\int_{\partial B(x_0,\frac{r}{2})} u \cdot \nu_i\, dx\Big| \cdot \frac{1}{\alpha(n)\cdot\left(\frac{r}{2}\right)^n} \le$$

$$\le \frac{1}{\cancel{\alpha(n)}\left(\frac{r}{2}\right)^n} \|u\|_{L^\infty(\partial B(x_0,\frac{r}{2}))}\, n\, \cancel{\alpha(n)} \left(\frac{r}{2}\right)^{n-1} =$$

$$= \frac{2n}{r} \|u\|_{L^\infty(\partial B(x_0,\frac{r}{2}))}$$



$\forall x \in \partial B(x_0, \frac{r}{2})$

$B(x, \frac{r}{2}) \subset B(x_0, r)$

$|u(x)| \le \dfrac{1}{\alpha(n)\left(\frac{r}{2}\right)^n} \cdot \|u\|_{L^1(B(x_0,r))}$ by $k=0$ estimate

$$\Rightarrow |D^\alpha u(x_0)| \le \frac{2n}{r\,\alpha(n)\left(\frac{r}{2}\right)^n} \|u\|_{L^1(B(x_0,r))} =$$

$$= \frac{2^{n+1} n}{r^{n+1}\alpha(n)} \|u\|_{L^1(B(x_0,r))}$$