Math 678.
Lecture 18

Separation of Variables.

(1) $\begin{cases} u_t - \Delta u = 0 & U \\ u = 0 & \partial U \\ u = g & t = 0 \end{cases}$ IBVP for heat eqn.

$u(x,t) = v(t) \cdot w(x)$

$u_t = v_t \cdot w$, $v \Delta w = \Delta u \Rightarrow$

$$0 = u_t - \Delta u = v'(t) w(x) - v(t) \Delta w$$

$$v'(t) w(x) = v(t) \Delta w$$

$$\frac{v'(t)}{v(t)} = \frac{\Delta w(x)}{w(x)} = \mu = \text{const}$$

$$\begin{cases} v'(t) = \mu v(t) \Rightarrow v(t) = Ce^{\mu t} \\ \Delta w = \mu w \end{cases}$$

Eigenvalue problem: $\begin{cases} -\Delta w = \lambda w & U \\ w = 0 & \partial U \end{cases}$

λ - eigenvalue, w - corresp. eigenvector (eigenfunction)

$\mu = -\lambda$

$u(x,t) = Ce^{\mu t} w(x) = Ce^{-\lambda t} w(x)$

If we need to satisfy $u(x,0) = g(x) \Rightarrow$ we get a condition of the form: $Cw(x) = g(x)$

More generally: $u(x,t) = \sum_{i=1}^{m} C_i e^{-\lambda_i t} w_i(x)$

or $u(x,t) = \sum_{i=1}^{\infty} C_i e^{-\lambda_i t} w_i(x)$

with C_i, λ_i, w_i chosen to satisfy $\sum_{i=1}^{\infty} C_i w_i(x) = g(x)$

We need to be able to find such values, and to make sure that the series converges.

(2) $u_t - \Delta(u^\gamma) = 0 \quad \gamma > 1 \quad u \geq 0$

$$u = v(t) w(x)$$
$$u_t = v'(t) \cdot w(x)$$
$$\Delta(u^\gamma) = \Delta(v^\gamma \cdot w^\gamma) = v^\gamma \cdot \Delta(w^\gamma)$$
$$\Rightarrow \quad 0 = u_t - \Delta(u^\gamma) = v' w - v^\gamma \Delta w^\gamma$$
$$\frac{v'}{v^\gamma} = \frac{\Delta w^\gamma}{w} = \mu = \text{const}$$

$$\begin{cases} v' = \mu v^\gamma \\ \Delta w^\gamma = \mu w \end{cases} \longrightarrow \frac{dv}{v^\gamma} = \mu\, dt$$

$$(-\cancel{\gamma}+1)\, v^{-\gamma+1} = \mu t + C$$
$$v^{1-\gamma} = (1-\gamma)\mu t + \underbrace{C(1-\gamma)}_{\tilde{C}}$$
$$v = \left((1-\gamma)\mu t + \tilde{C}\right)^{1/1-\gamma}$$

$$\Delta w^\gamma = \mu w$$

Look at $w = |x|^\alpha \Rightarrow w^\gamma = |x|^{\alpha\gamma}$

$$\frac{\partial w^\gamma}{\partial x_i} = \frac{\alpha\gamma |x|^{\alpha\gamma - 1} \cdot x_i}{|x|} = \alpha\gamma |x|^{\alpha\gamma-2} \cdot x_i$$
$$\frac{\partial^2 w^\gamma}{\partial x_i^2} = \alpha\gamma |x|^{\alpha\gamma-2} + \alpha\gamma x_i \cdot (\alpha\gamma - 2)|x|^{\alpha\gamma-3} \cdot \frac{x_i}{|x|}$$
$$\Delta w^\gamma = \sum_{i=1}^n \frac{\partial^2 w^\gamma}{\partial x_i^2} = \alpha\gamma |x|^{\alpha\gamma-2} \cdot n + \alpha\gamma(\alpha\gamma-2)|x|^{\alpha\gamma-3} \sum_{i=1}^n \frac{x_i^2}{|x|}$$
$$= \alpha\gamma|x|^{\alpha\gamma-2} n + \alpha\gamma(\alpha\gamma-2)|x|^{\alpha\gamma-2}$$
$$= \alpha\gamma|x|^{\alpha\gamma-2}(n + \alpha\gamma - 2)$$

$\Rightarrow$ In order to have $\Delta w^\gamma = \mu w$ we need

$$\alpha\gamma|x|^{\alpha\gamma-2}(n+\alpha\gamma-2) = \mu|x|^\alpha$$
$$\alpha\gamma - 2 = \alpha \Rightarrow \begin{cases} \alpha = \frac{2}{\gamma-1} \\ \mu = \alpha\gamma(n+\alpha\gamma-2) \end{cases}$$

$$\Rightarrow \quad u(x,t) = \left((1-\gamma)\mu t + \tilde{C}\right)^{1/1-\gamma} |x|^\alpha$$

$$(1-\delta)\mu t^* + \tilde{C} = 0 \Rightarrow t^* = \frac{\tilde{C}}{\mu(\delta-1)}$$ blow-up time (finite).

Separation of variables can be done in a different way: $u(x,t) = v(t) + w(x)$

(3) $u_t + H(Du) = 0$ in $\mathbb{R}^n$

Hamilton-Jacobi eqn, H – Hamiltonian

$$u(x,t) = v(t) + w(x)$$
$$u_t = v'(t)$$
$$0 = u_t + H(Du) = v'(t) + H(Dw(x))$$
$$-v'(t) = H(Dw(x)) = \mu$$

$$\begin{cases} v'(t) = -\mu \Rightarrow v(t) = -\mu t + C \\ H(Dw(x)) = \mu \end{cases} \Rightarrow u(x,t) = w(x) - \mu t + C$$

$$\left.\begin{aligned} w &= a\cdot x \\ \mu &= H(a) \end{aligned}\right\} \Rightarrow u(x,t) = a\cdot x - H(a)\cdot t + C$$

__Similarity solutions:__

$u(x,t) = v(x - \sigma t)$ — traveling wave

v – profile, σ – speed, $x \in \mathbb{R}$

$u(x,t) = v(y\cdot x - \sigma t)$ plane wave in $\mathbb{R}^n$

velocity $\frac{\sigma}{|y|}$

wave front is normal to y

$u(x,t) = e^{i(y\cdot x + \omega t)}$ ← plug this into a PDE.

(1) Heat equation:

$$u_t - \Delta u = i\omega u + |y|^2 u = (i\omega + |y|^2)u = 0$$
$$\omega = -\frac{|y|^2}{i} = i|y|^2$$
$$\Rightarrow u = e^{-|y|^2 t} \cdot e^{iy\cdot x}$$ – dissipates at $t \to \infty$.

② Wave equation:

$u_{tt} - \Delta u = (-\omega^2 + |y|^2)u = 0$

$\omega = \pm|y|$ - real

$\Rightarrow \quad u = e^{i(y\cdot x \pm |y| t)}$ no dissipation

$\cos(y\cdot x \pm |y|t)$
$\sin(y\cdot x \pm |y|t)$ — solve wave eqn

③ Airy's equation: $u_t + u_{xxx} = 0$

$\Rightarrow \quad u = e^{i(yx + \omega t)}, \quad n=1 \quad \Rightarrow$

$u_t + u_{xxx} = i(\omega - y^3)u = 0$

$\omega = y^3$

$u = e^{i(yx + y^3 t)}$ $\boxed{\text{dispersion}}$ of initial ~~frequency~~ profile.
Since ~~di~~ speed will vary depending on initial frequency.

Schrödinger's equation: $iu_t + \Delta u = 0$

$0 = iu_t + \Delta u \Rightarrow e^{i(y\cdot x - |y|^2 t)} = u(x,t)$

will lead to $\boxed{\text{dispersion}}$