Math 678. Homework 5 Solutions.

#1

Consider a subsolution of the heat equation $v_t - \Delta v \le 0$ in U_T.
(a) The proof follows the argument given in Theorem 3, p.53-54, with the exception being that $\phi'(r) \ge 0$, from where it follows that

$$v(x,t) \le \frac{1}{4r^n} \iint_{E(x,t;r} v(y,s) \frac{|x-y|^2}{(t-s)^2} dyds$$

for all heat balls $E(x,t;r) \subset U_T$.

(b) As in Theorem 4, it follows that $\max_{\bar{U}_T} v = \max_{U_T} v$. Indeed, suppose there is a point (x_0, t_0) in U_T where the function value is maximized on the entire closed domain $\bar{U}_T$. Then there is a sufficiently small heat ball around it where from the above we will have $M = v(x_0,t_0) \le \frac{1}{4r^n} \iint_{E(x,t;r} v(y,s) \frac{|x-y|^2}{(t-s)^2} dyds \le M$. Equality is only possible when $u \equiv M$ in the heat ball. Then we can cover the domain with such balls as in the proof of Theorem 4 and conclusion follows.

(c) Let $v = \phi(u)$ with ϕ being convex and u a solution to the heat equation. Notice that

$$\begin{aligned} &v_t = \phi'(u) u_t, \\ &\Delta v = \phi''(u) \textstyle\sum_{i=1}^n u_{x_i}^2 + \phi'(u) \Delta u = \phi''(u) \sum_{i=1}^n u_{x_i}^2 \end{aligned}$$

Since $\phi''(u) \ge 0$, we observe that $\Delta v \ge v_t$, so v is a subsolution.

(d) You can verify this directly, which is a tedious but straightforward calculation. Alternatively, you may notice that both $|\cdot|^2 and (\cdot)^2$ are smooth and convex, with Du and u being solutions to the heat equation, and apply the result of (c).

#2

Consider $u_{x_1}^2 u_{x_1x_1} + 2u_{x_1}u_{x_2}u_{x_1x_2} + u_{x_2}^2 u_{x_2x_2} = 0$ and separate the variables using $u(x,y) = v(x_1) + w(x_2)$. Plug this into the equation to get

$$(v')^2 v'' = -(w')^2 w'' = const = \lambda$$

This gives a system of ODEs:

$$\left\{ \begin{array}{l} (v')^2 v'' = \lambda \\ -(w')^2 w' = \lambda \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} (v')^3/3 = \lambda x_1 + C_1 \\ (w')^3/3 = -\lambda x_2 + C_2 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} v' = (3\lambda x_1 + C_1)^{1/3} \\ w' = (-3\lambda x_2 + C_2)^{1/3} \end{array} \right.$$

which after integration yields particular solutions of the form: $v(x_1) = x_1^{4/3}, w(x_2) = C_2 x_2^{4/3}$, where we made the easiest choice of the constants of integration. So a particular nontrivial solution of the original equation can be written for instance as $u(x_1, x_2) = x_1^{4/3} - x_2^{4/3}$.

#3

Plug in $v(x-\sigma t)$ into the equation to get $-\sigma v'(x-\sigma t) - v''(x-\sigma t) = f(v(x-\sigma t))$. which is equivalent to $v''v' + \sigma(v')^2 + f(v)v' = 0$. After integration we get (for $f(v) = -2v^3 + 3v^2 - v$):

$$(v')^2/2 + \sigma \int_{-\infty}^{s} (v')^2 - v^4/2 + v^3 - v^2/2 = C$$

Consider the limit when $s \to \infty$, then by employing the boundary conditions we get $\sigma \int_{-\infty}^{\infty} (v')^2 = C$ and then similarly for $s \to -\infty$, $0 = \sigma \int_{-\infty}^{-\infty} (v')^2 = C$. Hence $\sigma \int_{-\infty}^{\infty} (v')^2 = 0$, which implies $\sigma = 0$.

Hence we have to solve the following equation in v:

$$(v')^2/2 - v^4/2 + v^3 - v^2/2 = 0$$

Here is how:

$$\begin{aligned}
&(v')^2/2 = v^4/2 - v^3 + v^2/2 = v^2(v-1)^2/2 \\
&v' = \pm v(v-1) \\
&\Big(\frac{1}{v-1} - \frac{1}{v}\Big) dv = ds \\
&\ln\left|\frac{v-1}{v}\right| = \pm s + C
\end{aligned}$$

The only solution that satisfies the boundary conditions is $v = \dfrac{1}{1 - Ce^{-s}}$. Note that $v \to 1, s \to \infty$ and $v \to 0, s \to -\infty$ for any choice of C. This solution can be verified to satisfy $-v_{ss} = f(v)$ and is a degenerate case of a traveling wave.

#4

By Duhamel's Principle, the solution of the nonhomogeneous problem is obtained as

$$u(x,t) = \int_0^t u(x,t;s)ds$$

where $u(x,t;s)$ is the solution to the homogeneous BIVP problem on the interval $[0,1]$:

$$\begin{cases}
u_{tt} - u_{xx} = 0 \\
u(\cdot, s) = x^2(1-x) \\
u_t(\cdot, s) = x \\
u_x(0, \cdot) = 0 \\
u(1, \cdot) = 0
\end{cases}$$

This problem can be solved by separation of variables: $u(x,t) = v(t)w(x)$, which gives

$$\frac{v''}{v} = \frac{w''}{w} = -\lambda$$

With the boundary data given, the solution exists when $\lambda = -\pi^2(k+1/2)^2, w = \cos(\pi x/2 + \pi k x)$. The solution is then represented as a series

$$u(x,t;s) = \sum_{k=1}^{\infty} [A_k \cos(\pi t/2 + \pi k t)\cos(\pi x/2 + \pi k x) + B_k \sin(\pi t/2 + \pi k t)\cos(\pi x/2 + \pi k x)]$$

Plugging in the initial conditions, we get

$$\begin{aligned} A_k &= \tilde{A}_k \cos(\pi s/2 + \pi k s) - \tilde{B}_k \sin(\pi s/2 + \pi k s)/(\pi/2 + \pi k) \\ B_k &= \tilde{B}_k \sin(\pi s/2 + \pi k s) + \tilde{B}_k \cos(\pi s/2 + \pi k s)/(\pi/2 + \pi k) \end{aligned}$$

where $\tilde{A}_k, \tilde{B}_k$ are the Fourier coefficients for even extensions of the initial conditions:

$$\begin{aligned} \tilde{A}_k &= \textstyle\int_0^1 x^2(1-x)\cos(\pi x/2 + \pi k x)dx \\ \tilde{B}_k &= \textstyle\int_0^1 x\cos(\pi x/2 + \pi k x)dx \end{aligned}$$

so finally we get

$$u(x,t) = \int_0^t \sum_{k=1}^{\infty} [A_k \cos(\pi t/2 + \pi k t)\cos(\pi x/2 + \pi k x) + B_k \sin(\pi t/2 + \pi k t)\cos(\pi x/2 + \pi k x)]ds$$