Math 678. Homework 4 Solutions.

#1

(a) In order to get the general solution of $u_{xy} = 0$, use integration in one of the variables, e.g. in y first. This gives $u_x = f(x)$ and $u(x, y) = \int f(x)dx + G(y)$. Denote $F(x) = \int f(x)dx$ to get the conclusion: general solution of this equation is $u(x, t) = F(x) + G(y)$. Note: substitution can be used to verify this is indeed a solution, but by itself it does not prove this form gives all possible solutions of the original equation.

(b) Change of variables $\chi = x + t, \eta = x - t$ and use the fact that $x = (\chi + \eta)/2, t = (\chi - \eta)/2$ to get:

$$\begin{aligned} &u_\chi(\chi, \eta) = u_x x_\chi + u_t t_\chi = (u_x + u_t)/2, \\ &u_{\chi\eta}(\chi.\eta) = (u_{xx}x_\eta + u_{xt}t_\eta + u_{tt}t_\eta + u_{tx}x_\eta)/2 = (u_{xx} - u_{tt})/4 \end{aligned}$$

It follows that $u_{tt} - u_{xx} = 0$ iff $u_{\chi\eta} = 0$.

(c) Consider the wave equation in 1-d: $u_{tt} - u_{xx} = 0$ with initial conditions $u(x, 0) = g, u_t(x, 0) = h$. Due to (b), it is enough to consider equation $u_{\chi\eta} = 0$, which according to (a) has a general solution in the form $u(\chi, \eta) = F(\chi) + G(\eta)$ for arbitrary F and G. In other words, $u(x, t) = F(x + t) + G(x - t)$. Use initial conditions to deduce: $u(x, 0) = F(x) + G(x) = g(x)$ and $u_t(x, 0) = F_t(x) - G_t(x) = h(x)$. Integrating the second relation, we see that $F(x) - G(x) = 2F(x) - g(x) = \int_{-\infty}^x h(s)ds$. This means $F(x) = (g(x) + \int_{-\infty}^x h(s)ds)/2$ and $G(x) = g(x) - F(x) = (g(x) - \int_{-\infty}^x h(s)ds)/2$. Back to the general solution, we have $u(x, t) = \dfrac{1}{2}(g(x + t) + g(x - t)) + \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} h(s)ds$, which is the d'Alembert formula we know.

(d) Right-moving if $F \equiv 0$, i.e. $g'(x) = -h(x)$. Left-moving if $G \equiv 0$, i.e. $g'(x) = h(x)$.

#2

Consider the system

$$\begin{cases} u_t + u_x = d(v - u) \\ v_t - v_x = d(u - v) \end{cases}$$

First notice that $v_t - v_x = -(u_t + u_x)$. Differentiating the equations wrt t and then x, we get

$$\begin{aligned} &u_{tt} + u_{xt} = d(v_t - u_t), \\ &v_{tt} - v_{xt} = d(u_t - v_t) \\ &u_{tx} + u_{xx} = d(v_x - u_x) \\ &v_{tx} - v_{xx} = d(u_x - v_x) \end{aligned}$$

Subtracting the 3rd from the 1st equation, and adding 2nd and 4th equations, we get

$$u_{tt} - u_{xx} = d(v_t - u_t) - d(v_x - u_x) = d(v_t - v_x) - d(u_t - u_x) = -d(u_t + u_x) - d(u_t - u_x) = -2du_t$$
$$v_{tt} - v_{xx} = d(u_t - v_t) + d(u_x - v_x) = d(u_t + u_x) - d(v_t + v_x) = -d(v_t - v_x) - d(v_t + v_x) = -2dv_t$$

It follows that u and v both satisfy the equation $w_{tt} + 2dw_t - w_{xx} = 0$.

#3.

Consider change of variables

$$s = \frac{t - ax_1}{\sqrt{1-a^2}}$$
$$y_1 = \frac{x_1 - at}{\sqrt{1-a^2}}$$
$$y_2 = x_2, y_3 = x_3$$

In new coordinates, the wave equation should look like $u_{ss} = u_{y_1y_1} + u_{y_2y_2} + u_{y_3y_3}$, which is the statement we intend to justify. In this problem, we will start with the wave equation in original coordinates and rewrite it with the help of new variables. Let us compute the partial derivatives:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} + \frac{\partial u}{\partial y_1}\frac{\partial y_1}{\partial t} = \frac{1}{\sqrt{1-a^2}}\frac{\partial u}{\partial s} - \frac{a}{\sqrt{1-a^2}}\frac{\partial u}{\partial y_1}$$
$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x_1} + \frac{\partial u}{\partial y_1}\frac{\partial y_1}{\partial x_1} = -\frac{a}{\sqrt{1-a^2}}\frac{\partial u}{\partial s} + \frac{1}{\sqrt{1-a^2}}\frac{\partial u}{\partial y_1}$$

Hence

$$u_{tt} = \frac{1}{1-a^2}(u_{ss} - 2au_{sy_1} + a^2u_{y_1y_1})$$
$$u_{x_1x_1} = \frac{1}{1-a^2}(u_{y_1y_1} - 2au_{sy_1} + a^2u_{ss})$$
$$u_{x_2x_2} = u_{y_2y_2}$$
$$u_{x_3x_3} = u_{y_3y_3}$$

It follows that $0 = u_{tt} - \Delta u = u_{ss} - u_{y_1y_1} - u_{y_2y_2} - u_{y_3y_3}$, QED.