Math 678. Homework 1 Solutions.

#2, p.12

Think about circles denoting the partial derivative, with m consecutive circles representing an m-th order derivative in the corresponding variable. Dividers are placed between variables, and can be separated by m circles, with $m \geq 0$. If $m = 0$, no derivative is taken in the corresponding variable. This gives a one-to-one correspondence between a pattern of k circles with $n-1$ dividers and a k-th order partial derivative of a function of n variables.

To give a simple example, let $n = 5, k = 3$. The derivative $\dfrac{\partial^3}{\partial x_1 \partial x_2 \partial x_5}$ can be represented as $\circ|\circ|||\circ$. The derivative $\dfrac{\partial^3}{\partial x_1 \partial x_4^2}$ can be represented as $\circ|||\circ\circ|$ etc. The number of ways of inserting $n-1$ dividers within a row of k circles in this way to represent each partial derivative of k-th order is given by

$$\left(\begin{array}{c} n+k-1 \\ k \end{array} \right) = \left(\begin{array}{c} n+k-1 \\ n-1 \end{array} \right)$$

Equality follows from the symmetry in the definition of the combination.

#4, p.13

To show (Leibniz's formula):

$$D^\alpha(uv) = \sum_{\beta \leq \alpha} \left(\begin{array}{c} \alpha \\ \beta \end{array} \right) D^\beta u D^{\alpha-\beta} v$$

The proof goes by induction. First, notice it is obviously true for $|\alpha| = 0$ (equivalence) and $|\alpha| = 1$ (product rule). Suppose it holds for all $|\beta| \leq p$ and let $|\alpha| = p+1$, i.e. $\alpha = \beta + \gamma$ for some $|\beta| = p$, $|\gamma| = 1$. All Greek letter represent a multiindex notation. Using the induction hypothesis and product rule,

$$D^\alpha(uv) = D^\gamma(D^\beta(uv)) =$$
$$D^\gamma \left(\sum_{\sigma \leq \beta} \left(\begin{array}{c} \beta \\ \sigma \end{array} \right) D^\sigma u D^{\beta-\sigma} v \right) =$$
$$\sum_{\sigma \leq \beta} \left(\begin{array}{c} \beta \\ \sigma \end{array} \right) D^\gamma \left(D^\sigma u D^{\beta-\sigma} v \right) =$$
$$\sum_{\sigma \leq \beta} \left(\begin{array}{c} \beta \\ \sigma \end{array} \right) \left(D^\omega u D^{\alpha-\omega} v + D^\sigma u D^{\alpha-\sigma} v \right)$$

where $\omega = \sigma + \gamma, \beta - \sigma = \alpha - \omega$. Now we can break this into two sums, and change variables in the second one, noticing that $\sigma \leq \beta$ is equivalent to $\omega \leq \alpha$. This results in

$$\sum_{\sigma \leq \beta} \left(\begin{array}{c} \beta \\ \sigma \end{array} \right) D^\omega u D^{\alpha-\omega} v + \sum_{\omega \leq \beta} \left(\begin{array}{c} \beta \\ \omega \end{array} \right) D^\omega u D^{\alpha-\omega} v =$$
$$\sum_{\omega \leq \alpha} \left(\begin{array}{c} \alpha-\gamma \\ \omega-\gamma \end{array} \right) D^\omega u D^{\alpha-\omega} v + \sum_{\omega \leq \alpha} \left(\begin{array}{c} \alpha-\gamma \\ \omega \end{array} \right) D^\omega u D^{\alpha-\omega} v$$

The sought conclusion follows by employing Pascal's formula (valid for $|\gamma| = 1$):

$$\begin{pmatrix} \alpha - \gamma \\ \omega - \gamma \end{pmatrix} + \begin{pmatrix} \alpha - \gamma \\ \omega \end{pmatrix} = \begin{pmatrix} \alpha \\ \omega \end{pmatrix}$$

#1, p.85

$$\begin{cases} u_t + b \cdot Du + cu = 0, & t > 0 \\ u = g, & t = 0 \end{cases}$$

Following the proof of the transport equation, we introduce $z(s) = u(x + sb, t + s), s \in \mathbb{R}$. Then it is easy to see that $\dot{z}(s) = -cz(s)$, which means $z(s) = Ae^{-cs}$. Since $z(0) = u(x,t)$, $A = u(x,t)$. Now notice that $z(-t) = u(x - tb, 0) = g(x - tb)$, so that $g(x - tb) = u(x,t)e^{ct}$, which means that

$$u(x,t) = e^{-ct} g(x - tb).$$

#2 p.85

To show: Laplace equation is rotation invariant.

Denote $v(x) = u(Ox)$ and let $y = Ox$. By chain rule, for any index i we have

$$\frac{\partial v}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial u}{\partial y_j} \frac{\partial y_j}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial u}{\partial y_j} O_{ji}$$

Since $O^T O = O O^T = I$ due to orthogonality,

$$\sum_{i=1}^{n} O_{ki} O_{ji} = (OO^T)_{kj} = \delta_{ij},$$

so that it is equal to 1 when $k = j$ and is zero otherwise.

By interchanging the order of summation, we easily see that

$$\Delta v = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} = \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial y_j \partial y_k} O_{ki} O_{ji} = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

The conclusion immediately follows, since u satisfies the Laplace equation.

We could have written the same argument in a vector form, by noticing that $D_x v = O^T D_y u$ and $\Delta v = D_x v \cdot D_x v = (O^T D_y u, O^T D_y u) = (D_y u, OO^T D_y u) = (D_y u, D_y u) = \Delta u = 0$.