Periodic orbits, limit cycles.

\[ x = f(x) \]

**Def.** A cycle is any closed solution curve which is not an equilibrium.

→ **stable** if \( \forall \varepsilon > 0 \exists U_r(\Gamma) \text{ s.t.} \)

\[ \forall x \in U \quad d(\Gamma_x, \Gamma) < \varepsilon \]

\[ \forall x \in U \quad \lim_{t \to 0} d(\psi(t, x), \Gamma) = 0. \]

→ **unstable** if it's not stable

→ **asympt. stable** if it is stable and

\[ \lim_{t \to \infty} \psi(t, x), \Gamma = 0. \]

Periodic orbit: \( \psi(t+T, x) = \psi(t, x) \)

Center for linear system:

In general, \( T \to \infty \) as you go away from equilibrium.

\[ \Gamma : x = f(t), \quad 0 \leq t \leq T \]

asympt. stable \( \iff \int_0^T \Delta \cdot f(t(x)) \, dt \leq 0 \]

In 2D, asympt. stable \( \iff \int_0^T \Delta \cdot f(t(x)) \, dt < 0 \)

\( \omega \)-limit cycle: asympt. stable cycle is an attractor.
Local invariant manifolds:
\[ S = \{ x \in N | d(\psi_t(x), r) \to 0 \text{ as } t \to \infty, \quad \psi_t(x) \in N, \quad t \geq 0 \} \]
stable
\[ U = \{ x \in N | d(\psi_t(x), r) \to 0 \text{ as } t \to -\infty, \quad \psi_t(x) \in N, \quad t \leq 0 \} \]
unstable

Global manifolds:
\[ W^s(\Gamma) = \bigcup_{t \geq 0} \psi_t(S(\Gamma)) \]
\[ W^u(\Gamma) = \bigcup_{t \geq 0} \psi_t(U(\Gamma)) \]

Ex.
\[ \begin{align*}
    \dot{x} &= -y + x (1 - x^2 - y^2) \\
    \dot{y} &= x + y (1 - x^2 - y^2) \\
    \dot{z} &= z
\end{align*} \]

Invariant sets:
\{z-axis\} \cup \{x^2 + y^2 = 1\} \cup \{xy\text{-plane}\}
\[ \Gamma: \quad x(t) = (\cos t, \sin t, 0) \]\n\[ W^u(\Gamma) = \{ x^2 + y^2 = 1 \} \]

Cylinder \{ x^2 + y^2 = 1 \} - attracting set
periodic orbit of saddle type

In 2D (Planar system):

**Def.** Limit cycle in 2D is a cycle that is either an \( W \)-limit set of or \( L \)-limit set for some trajectory other than \( \Gamma \).
A limit cycle is stable (unstable) if it
is an ω-limit for all traj. in some Nₖ(Γ),
semi-stable if it's an ω-limit for
some traj. and an o-limit for some others.

Ex. \[
\begin{align*}
\dot{x} &= -y + x(x^2 + y^2) \sin \frac{1}{\sqrt{x^2 + y^2}} \\
\dot{y} &= x + y(x^2 + y^2) \sin \frac{1}{\sqrt{x^2 + y^2}}
\end{align*}
\]
for \(x^2 + y^2 > 0\)
\(x = y = 0, \ x^2 + y^2 = 0\).

\[S^1 = \{ r \sin \frac{1}{r} \} \quad \Gamma_n: \ r = \frac{1}{\pi n}, n \in \mathbb{Z} \]
\(\Theta = 1\)
\(\Gamma_{2n} - \text{stable} \)
\(\Gamma_{2n+1} - \text{unstable} \)

Thm. If a traj. in \(\text{ext}(Γ)\) has \(Γ\) as a
\(w\)-limit cycle
\(w\)-limit, then all traj. in \(N_Γ(Γ) \cap \text{ext}(Γ)\)
have \(Γ\) as a \(w\)-limit.

\(Γ\)

If \(y(t, x) \in \text{ext}(Γ)\) spirals toward \(Γ\),
all of them spiral toward \(Γ\), and
i.e. intersect a straight line \(L \perp Γ\)
the int. many times at \(t = t_n, n \to \infty\).

Thm. (Dulac). For any bond region in \(R^2\),
any analytic system has a fin. number
of limit cycles.
Homoclinic orbits

\[ w(\Gamma) = \{ p \} \]
\[ \alpha(\Gamma) = \{ p \} \]
\[ p = q \]

\[ \begin{cases} 
  x = y \\
  y = x + x^2
\end{cases} \quad (1) \]

Hamiltonian

\[ H = \frac{x^2}{2} + \frac{y^2}{2} + \frac{k^3}{3} \]

\[ y^2 - x^2 - \frac{2}{3} x^3 = c \]

\[ \Gamma : \{ c = 0 \} \subset W^s(0) \cup W^u(0) \]

Homoclinic orbit : contained in both \( W^s(p), W^u(q) \)

Heteroclinic orbit : same but \( p \neq q \).

Flow on \( \Gamma \cup \{ 0 \} \) is separatrix cycle in (1)
Flow on \( \Gamma \cup \{ 0 \} \cup \{ p \} \cup \{ q \} \) - separatrix cycle in (2)

Compound separatrix cycles : union of

compatibly oriented separatrix cycles.

Undamped pendulum

\[ x + \sin x = 0 \quad (2) \]

Newtonian