\[ \dot{x} = f(x, \mu) \quad f \in C^1(E \times V), \quad E \subset \mathbb{R}^n, \quad V \subset \mathbb{R} \]

**Def.** \( f(x, \mu) \) is called structurally stable at \( x \in E \times V \) if \( \exists \varepsilon > 0 \) s.t. \( \forall \varepsilon \in C^1(E \times V) \) with \( \|f - g\| < \varepsilon \)

\[ \implies f, g \text{ are top. conjugate (i.e. there is an H-homeo s.t. } H \text{ maps trajectories of } \dot{x} = f \text{ onto traj. of } \dot{x} = g \text{, preserving orientation).} \]

\[ \implies \text{In this case, } \dot{x} = f(x, \mu) \text{ is called structurally stable, otherwise } \neq \text{ structurally unstable.} \]

**Def.** \( \mu = \mu_0 \) is a **bifurcation point** if \( f(x, \mu) \) is structurally unstable at \( \mu = \mu_0 \).

Today: 1D non-hyperbolic crit. pt. bifurcations.

**Ex. 1** \( \dot{x} = \mu - x^2 \) (Saddle-node bifurcation)

\[ (\text{or } \dot{x} = \mu + x^2) \quad Df(x, \mu) = \frac{\partial f}{\partial x}(x, \mu) = -2x \]

1) \( \mu = 0 \implies x = 0 \) only crit. pt. \( Df(0, 0) = 0 \) \( W^s = (0, 0) \)

2) \( \mu > 0 \implies x = \pm \sqrt{\mu} \) - crit. pts \( Df(\pm \sqrt{\mu}, \mu) = \mp 2\sqrt{\mu} \)

\[ x = \sqrt{\mu} \text{ - stable} \]

\[ x = -\sqrt{\mu} \text{ - unstable} \]

3) \( \mu < 0 \implies \text{no crit. pts} \)

"blue-skies" fold bifurcation

\( W^s = (-\sqrt{\mu}, 0) \)

\( W^u = (0, -\sqrt{\mu}) \)
Bifurcation diagram:

\[
\begin{align*}
\dot{x} &= \mu x - x^2 \\
Df(x, \mu) &= \frac{\partial f(x, \mu)}{\partial x} = \mu - 2x
\end{align*}
\]

Ex. 2

1) \( \mu = 0 \Rightarrow x_0 = 0 \) only crit. pt. \( Df(0, 0) = 0 \)
   non-hyperbolic pt
   \( f \) - structurally unstable
   \( \mu = 0 \) - bifurcation value

2) \( \mu \neq 0 \)
   \( x_0 = 0 \) \( Df(0, \mu) = \mu \)
   \( \mu = x_0 \) \( Df(\mu, \mu) = -\mu \)

   \( \mu > 0 \Rightarrow x = 0 \) is unstable
   \( x = \mu \) stable

   \( \mu < 0 \Rightarrow x = 0 \) not stable
   \( x = \mu \) unstable

Unstable point becomes stable and vice versa.
Transcritical bifurcation.
Ex. 3 \( x = \mu x - x^3 \) (OR \( \dot{x} = \mu x + x^3 \))

- **Supercritical pitchfork bifurcation**
- **Subcritical pitchfork bifurcation**

\( Df = \mu - 3x^2 \)

1. \( \mu = 0 \)
   - \( x_0 = 0 \) only crit. pt. \( Df(0,0) = 0 \)
   - Non-hyperbolic equilibrium
   - \( f(x) = -x^3 \) structurally unstable

2. \( \mu > 0 \)
   - \( \mu = x^2 \) \( x_0 = 0 \) \( Df(0,\mu) = \mu \)
   - \( x = 0 \) \( x = \pm \sqrt{\mu} \)
   - \( DF(\pm \sqrt{\mu}, \mu) = -2\mu \)
   - \( x_0 = 0 \) unstable
   - \( x = \pm \sqrt{\mu} \) - stable

3. \( \mu < 0 \)
   - \( x_0 = 0 \) \( Df(0,\mu) = \mu < 0 \)
   - \( x \) - pitchfork

In general, if \( x = f(x, \mu) \), \( x \in \mathbb{R} \) satisfies

1. \( f(-x, \mu) = -f(x, \mu) \)
2. \( f_x(0, \mu_0) = f_{xx}(0, \mu_0) = 0 \), \( f_{xxx}(0, \mu_0) \neq 0 \)
3. \( f_{\mu}(0, \mu_0) = 0 \), \( f_{\mu x}(0, \mu_0) \neq 0 \)

\( \Rightarrow \) if \( f_{xxx}(0, \mu_0) > 0 \) = **Subcritical pitchfork bifurcation**
\( f_{xxx}(0, \mu_0) < 0 \) = **Supercritical pitchfork bifurcation**
If, \( Df(0,0) = D^2f(0,0) = \ldots = D^{(m-1)}f(0,0) = 0 \)
\[ \frac{\partial f}{\partial x} \]
but \( D^m f(0,0) \neq 0 \) \Rightarrow \((0,0)\) - crit. point of
multiplicity = m
At most m points can bifurcate from \((0,0)\),
and there is such a bifurcation value.

\[ \mu = 0 \] - bifurcation value corresp. to

\((0,0)\) being a crit. pt. of multiplicity

\[ m = 2 \text{ in Ex 1-2} \]
\[ m = 3 \text{ in Ex 3} \]

If \( f(x_0, \mu_0) = Df(x_0, \mu_0) = 0 \)
nonhyperbolic crit. pt \( x_0 \) has a bifurcation of the type which depends on \( \frac{D^m f(x_0, \mu_0)}{\partial x \partial \mu} \), \( m \geq 2 \)

Theorem (Sotomayor) - Higher-dim case - \( \mathbb{R}^n \)

1) \( f(x_0, \mu_0) = 0 \)
2) \( A = Df(x_0, \mu_0) \) has \( \lambda_1 = 0 \) as a simple eigenvalue
and the rest of the eigenvalues look like
\[ \text{Re} (\lambda_2, \ldots \lambda_{k+1}) < 0 \]
\[ \text{Re} (\lambda_{k+2}, \ldots \lambda_{n}) > 0 \]
\[ 0 < k < n-k-1 \]
3) \( \nu \) - eig. vec. of \( A \) corresp. to \( \lambda_i = 0 \)
\( \psi \) - eig. vec. of \( A^T \) corresp. to \( \lambda_i = 0 \)
4) \( w^T f_{\mu}(x_0, \mu_0) \neq 0 \)
\[ w^T [D^2f(x_0, \mu_0)(\nu, \nu)] \neq 0 \]

\( \Rightarrow \exists \) smooth curve of equilibria in \( \mathbb{R}^{n \times \mathbb{R}} \)
passing through \((x_0, \mu_0)\) tangent to \( \mathbb{R}^{n \times \mathbb{R}} \)
defining a saddle-node bifurcation.
Conditions should be replaced with

\( w^T f_\mu(x_0, \mu_0) = 0 \)

\( w^T [Df_\mu(x_0, \mu_0)] v \neq 0, \ w^T [D^2f(x_0, \mu_0)(v, v)] \neq 0 \)

in transcritical bifurcation case

Or with

\( w^T f_\mu(x_0, \mu_0) = 0, \ w^T [Df_\mu(x_0, \mu_0)] v \neq 0 \)

\( w^T [D^2f(x_0, \mu_0)(v, v)] = 0, \ w^T [D^3f(x_0, \mu_0)(v, v, v)] \neq 0 \)

in pitchfork bifurcation case

Furthermore, conditions define a set of \( C^\infty \) functions that is open, dense in the Banach space of all \( C^\infty \) vector fields with 1 parameter having equilibrium at \( x_0 \) and a simple eigenvalue at \( \lambda = 0 \).

So saddle-node bifurcation is "generic" (can be obtained by perturbation).