Global stable (unstable) manifold is defined as
\[ W^s(0) = U \Phi_t(0), \quad W^u(0) = U \Phi_t(0) \]
respectively, forward in time.

\[ W^s, W^u - \text{unique, invariant under } \Phi_t. \]

\[ \forall x \in W^s(0), \quad \lim_{t \to -\infty} \Phi_t(x) = 0 \]
\[ \forall x \in W^u(0), \quad \lim_{t \to +\infty} \Phi_t(x) = 0. \]

**Ex.** \[ \begin{cases} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= x_2 + x_1^2 \end{cases} \quad x = Ax, \quad A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \] saddle.

\[ \begin{cases} x_1 &= x_1^0 e^{-t} \\ x_2 &= e^t (x_2^0 + \frac{1}{2} (x_1^0)^2) - \frac{1}{2} e^{-2t} (x_1^0)^2 \end{cases} \]

\[ W^u(0) = \{ x_1^0 = 0 \} = \{ x_2 - \text{axis} \} \]

\[ W^s(0) = \{ x_2 + \frac{1}{2} (x_1^0)^2 = 0 \} = \{ x_2 = -\frac{1}{3} x_1^0 \} \]

**Ex.** \[ \begin{cases} \dot{x}_1 &= x_1^2 \\ \dot{x}_2 &= -x_2 \end{cases} \quad x = Ax, \quad A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ E^s = \{ x_2 - \text{axis} \} \]

\[ E^c = \{ x_1 - \text{axis} \} \]

\[ W^c \text{ is composed of many curves tangent to } E^c \text{ coinciding with } x_1 \text{-axis for } x_1 > 0. \]

\[ W^c \text{ is not unique.} \]

analytical \( W^c \) is unique and coincides with \( E^c \).
Corollary of SMT:

Under the conditions of SMT

Let $S, U$ be stable and unstable manifolds of $\dot{x} = f(x)$ at $0$, $\lambda_i$: eig. values of $Df(0)$.

$\text{Re}(\lambda_j) < -\beta < 0 < \beta < \text{Re}(\lambda_m) ; j = 1, \ldots, k \quad m = k + 1, \ldots n$

$\Rightarrow \quad \forall \epsilon > 0 \ \exists \delta > 0 \ \text{s.t. } \text{if } x_0 \in N_{\delta}(0) \cap S$

$\text{then } |y^s_t(x_0)| \leq \epsilon e^{-\beta t} \quad \forall t \geq 0$

$\forall \epsilon > 0 \ \exists \delta > 0 \ \text{s.t. } \text{if } x_0 \in N_{\delta}(0) \cap U$

$\text{then } |y^u_t(x_0)| \leq \epsilon e^{\beta t} \quad \forall t \leq 0$

Thm. (CMT) Center Manifold Thm.

$f \in C^r(\mathbb{R})$, $EC = \text{-open}$, $f(0) = 0$

$Df(0)$ has $\lambda_i$: eig. values s.t.

$\text{Re}(\lambda_i) < 0 \ \text{ for } i = 1, \ldots k$

$\text{Re}(\lambda_i) > 0 \ \text{ for } i = k + 1, \ldots k + j$

$m = n - k - j$

$\text{Re}(\lambda_i) = 0 \ \text{ for } i = k + j + 1, \ldots n$

$\Rightarrow$ There exists an $m$-dim manifold $W^c(0)$

of class $C^r$ tangent to $E^c$ at $0$

Such a center manifold

$W^s(0), W^u(0)$ - tangent to $E^s, E^u$ resp.

Moreover, $W^u, W^s, W^c$ - $y^u_t$ - invariant.

Comment: $W^u, W^s, W^c$ can fail to be embedded manifolds in $\mathbb{R}^n$.

Ex. $\frac{dx}{dt} = x^2, \quad y = xe^{1/2} \rightarrow \text{family of center manifolds.}$
Thm. (Hartman–Groban Thm.)

1. \( \dot{x} = f(x) \)
2. \( \dot{x} = Ax, \ A = Df(x) \)

Equilibrium \( x_0 \) translated to the origin if \( \exists \psi \)-homeomorphism \( U \rightarrow V \) open \( \rightarrow \) open \( \psi_{03} \in U, \ \psi_{03} \in V \)

which maps trajectories of (1) into in \( U \)
into trajectories of (2) in \( V \), and preserves their time-orientation.

If \( \psi \) preserves parameterization by time,
then (1), (2) are called **top. conjugate**
in a neighborhood of \( x_0 \).

**Ex.** \( A = \begin{bmatrix} -1 & -3 \\ -3 & -1 \end{bmatrix} \) \( B = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \) \( \dot{x} = Ax \) \( \dot{y} = By \)

\( H(x) = Rx \) s.t. \( R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix} \) rotation by \( 45^\circ \)

\( H' : \begin{array}{c} A : \\ H \end{array} \Rightarrow \begin{array}{c} B = RAR^{-1} \\ y = R \end{array} \Rightarrow \begin{array}{c} \dot{y} = RAR^{-1} \dot{y} = By \\ x(t) = e^{At} x_0 \Rightarrow y(t) = Re^{At} x_0 = e^{At} y_0 \end{array} \)
Hartman–Grobman Theorem

\[ \mathbb{R}^n \text{-open } \Rightarrow \text{ EE } \]
\[ f \in C^1(\mathbb{R}) \quad \text{solution of } \dot{x} = f(x), \quad f(0) = 0 \]
\[ A = Df(0). \]

If \( A \) has no eigenvalue with \( \text{Re}(\lambda) = 0 \), then:

\[ \exists \text{ H-homeo: } U \rightarrow V \text{ s.t. } \]
\[ \forall x \in U \exists I \subset \mathbb{R} \text{ s.t. } \forall t \in I, \quad H \circ \psi_t(x) = e^{\lambda t} H(x) \]

(\( H \) maps trajectories of \( \dot{x} = f(x) \) near \( x_0 \) onto trajectories of \( \dot{x} = Ax \) near \( x_0 \) and preserves parameterization in time).

**Ex.**

\[ \begin{cases} \dot{x}_1 = -x_1 - \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \\ \dot{x}_2 = -x_2 + \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \end{cases} \quad \Rightarrow \begin{cases} \dot{y}_1 = -y_1 \\ \dot{y}_2 = -y_2 \end{cases} \]

(0, 0) - spiral

H-homeo but not a diffeomorphism

Reason: \( f \in C^1(\mathbb{R}) \) but not \( C^2(\mathbb{R}) \)

\( f \in C^2(\mathbb{R}) \).

*Then (Hartman)*

\[ \mathbb{R}^n \text{-open } \Rightarrow \text{ EE } \]
\[ f \in C^2(\mathbb{R}), \quad \text{Re} \lambda_i \neq 0 \text{ for } Df(0) \text{ eigenvalues} \]

\[ \Rightarrow \exists C^1 \text{-diffeomorphism } H \text{ mapping} \]

trajectories of (1) onto (2) and back.
\[ H : U \rightarrow V, \]
\[ \text{open} \quad \text{open} \]
\[ \forall x \in U \exists \xi(x) \in \mathbb{R} \text{ s.t.} \]
\[ \forall x \in U \forall t \in I \quad H \cdot \xi_t(x) = e^{At} H(\xi). \]