Part I. Exercises are taken from "Differential Equations and Dynamical Systems" by Perko, 3rd edition.

Problem Set 2.11: # 2
(a) \( \dot{x} = x^2, \dot{y} = y \). Expansion of the function \( \psi = p_2(x, \phi) \) is a nbhd of the origin has the form \( \psi = a_m x^m + \ldots \) with \( m \geq 2, a_m \neq 0 \). \( m = 2 \), so the origin is a saddle node.
(b) \( \dot{x} = x^2 + y^2, \dot{y} = y - y^2 \). Same as above.
(c) \( \dot{x} = y^2 + x^3, \dot{y} = y - x^2 \). \( m = 3, a_m = 1 \), so unstable node.

Problem Set 2.12: # 1
\( \dot{x} = x^2, \dot{y} = -y \). Linearization gives \( \lambda = 0, -1 \), so \( c = 1, s = 1, n = 2 \). Take \( h(x) = ax^2 + bx^3 + \ldots \). Plug into \( Dh(x)[Cx + F(x, h(x))] - Ph(x) - G(x, h(x)) = 0 \), we get \( h(x) = 0, \forall x \). \( W^c(0) = E^c \). By direct computation,
\[
  h(x, c) = \begin{cases} 
  0, & \text{for } x \geq 0 \\
  ce^{1/x}, & \text{for } x < 0 
  \end{cases}
\]
satisfies the above equation, so it defines a \( C^\infty \) center manifold for the system.

Problem Set 2.12: # 2
\[
\begin{align*}
  \dot{x} &= y \\
  \dot{y} &= -y + ax^2 + xy
\end{align*}
\]
The eigenvalues of the linearized system are \( \lambda = 0, -1 \). The Jordan form is
\[
  J = \begin{bmatrix} 0 & 0 \\
  0 & -1 \end{bmatrix}
\]
So we have \( C = 0, P = [-1], F(x, y) = 0, G(x, y) = ax^2 + xy \). Expand \( h(x) = ax^2 + \ldots \) in the equation for center manifold to get the system’s flow on the center manifold as \( \dot{x} = x^2 + O(|x|^3) \).

Part II, #1
\[
\begin{align*}
  \dot{x} &= -xy \\
  \dot{y} &= -y + x^2 - 2y^2
\end{align*}
\]
We get $\lambda = 0, -1$, $C = 0$, $P = [-1]$, $F(x,y) = -xy$, $G(x,y) = x^2 - 2y^2$. Let $h(x) = ax^2 + O(x^3)$. Then by the Center Manifold Theorem, $-4a^2x^3 + ax^2 - x^2 + 2y^2 = 0$, which means that $a = 1$. Therefore $\dot{x} = -x^3 + O(x^4)$.

Part II, #2
(a) $E^s = \{y-\text{axis}\}$, $E^c = \{x_1, x_2 - \text{plane}\}$.
(b) By CMT, $h(x) = ax_1^2 + bx_1x_2 + cx_2^2$ should satisfy $a = -1, b = 0, c = -1$, i.e. $C = \{y = h(x) = -x_1^2 - x_2^2 + (|x|^3)\}$.
(c) The equation of the flow on the center manifold is then

\[
\begin{align*}
\dot{x}_1 &= -x_2 - x_1x_2^2 - x_1^3 + O(|x|^3) \\
\dot{x}_2 &= x_1 - x_2^3 - x_1^2x_2 + O(|x|^3)
\end{align*}
\]

(d) In polar coordinates $\dot{r} = -r^3 + (|r|^3), \dot{\theta} = 1 + O(|\theta|^3)$