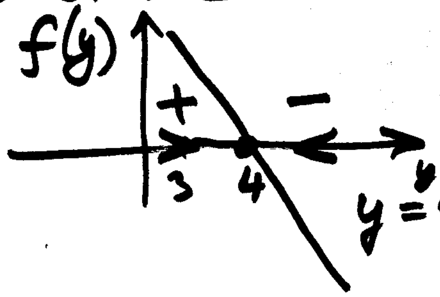
Math 214.002

Lecture 9.

Autonomous systems (equations)

Ex. 1 $y' = 4 - y$ classify equilibria as stable or unstable

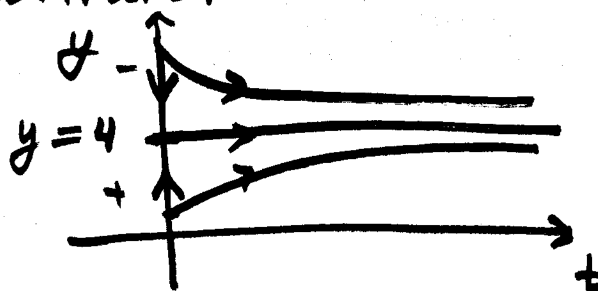
1) $f(y) = 4 - y$
zeros: $y = 4$



phase line

$y = 4$ asymp. stable equilibrium

2) Phase portrait:



3) Test for inflection points:

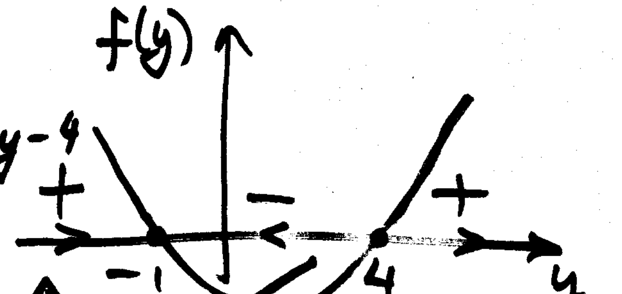
$f'(y) = 0$

$f'(y) = -1 \neq 0 \Rightarrow$ no inflection points.

$y'' = f(y) \cdot \underline{\underline{f'(y)}}$ Inflection pts occur only when $f'(y) = 0$.

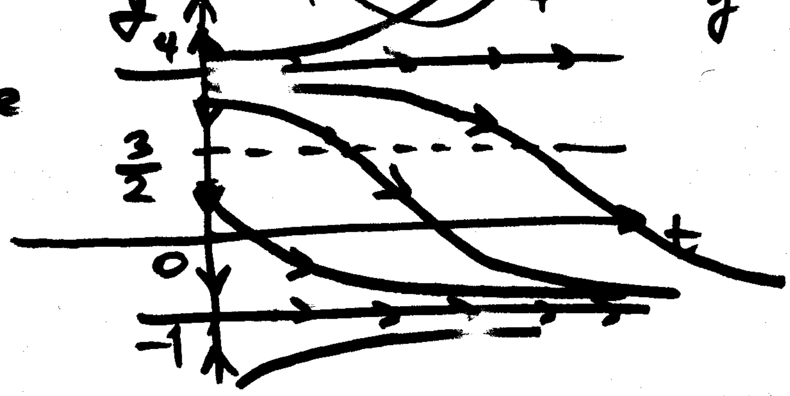
Ex. 2 $y' = (y+1)(y-4)$

1) $f(y) = (y+1)(y-4) = y^2 - 3y - 4$
zeros: $y = -1, 4$

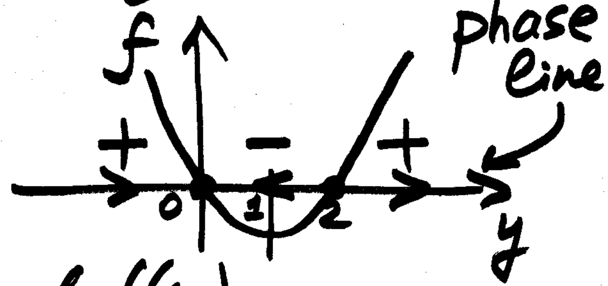


2) $y = -1$ stable
$y = 4$ unstable

3) $f'(y) = 2y - 3 = 0$
$y = \frac{3}{2}$ inflect. pt.



Example. $\frac{dy}{dt} = 2y(y-2)$ $\quad \lim_{t\to\infty} y(t) = ?$



1) Graph $f(y)$

$$f(y) = 2y(y-2)$$

2) Find zeros $\quad \boxed{\begin{array}{l} y=0 \\ y=2 \end{array}}$ > zeros of $f(y)$ (equilibrium solutions of DE)

2) Find $f'(y)=0$ critical points

$f' = 4y - 4 = 0$ $\quad$ (inflection points in DE).

$\boxed{y=1}$

$$\frac{dy}{dt} = f(y) \Rightarrow y'' = f \cdot f'$$

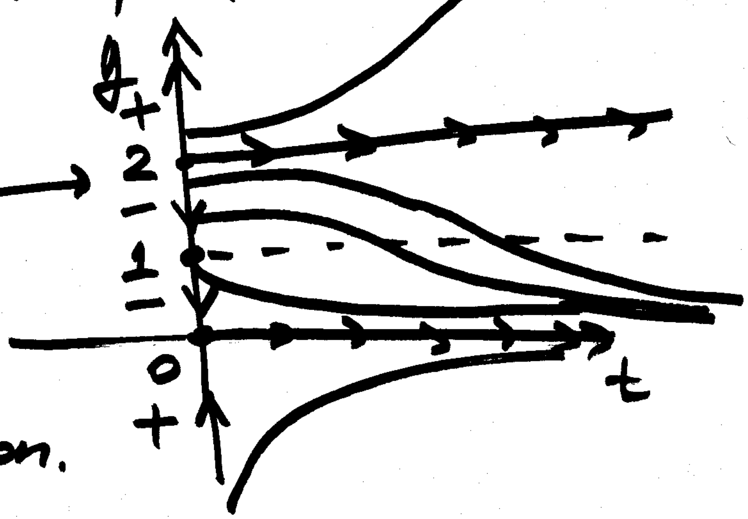
3) Determine intervals of $f>0$, $f'>0$

	f	f'	$f \cdot f'$
$y<0$	+	−	−
$0<y<1$	−	−	+
$1<y<2$	−	+	−
$y>2$	+	+	+

concave down

concave up

4) Phase portrait:



$y=2$ unstable Solution

$y=0$ asymptotically stable solution.

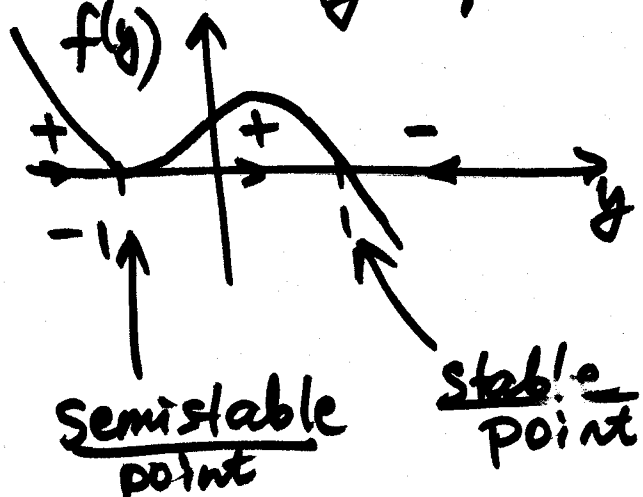
	f	$f' = 2y-3$	$f \cdot f'$
$y < -1$	$+$	$-$	$-$
$-1 < y < \frac{3}{2}$	$-$	$-$	$+$
$\frac{3}{2} < y < 4$	$-$	$+$	$-$
$y > 4$	$+$	$+$	$+$

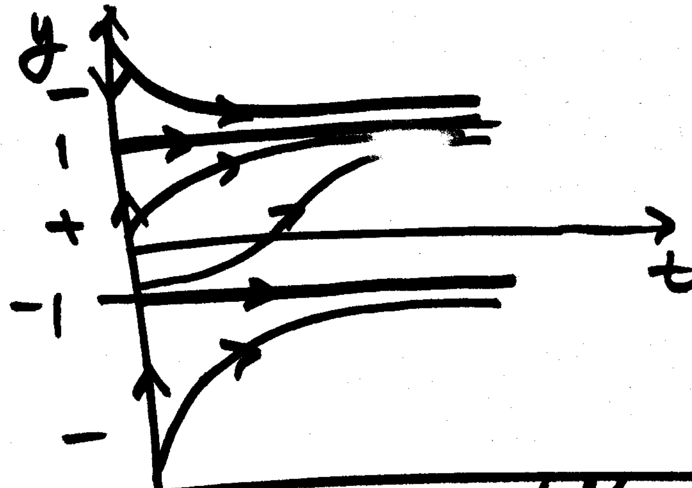
concave down

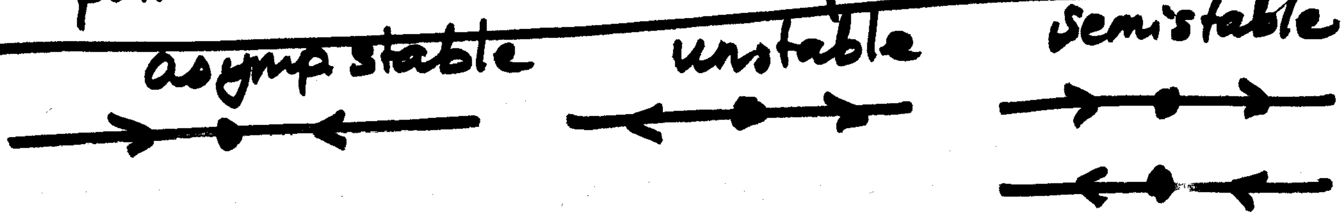
concave up

Ex. 3 $y' = -(y+1)^2(y-1)$

classify equilibria as stable or unstable



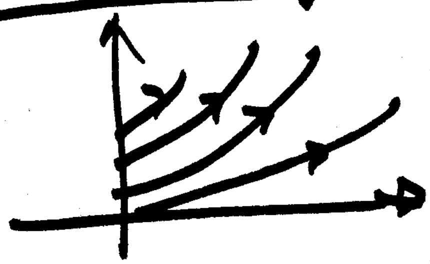




Population dynamics

(1) Regular exp. growth: $y' = ry,\ y(0) = y_0$

$y(t) = ce^{rt} = y_0 e^{rt}$

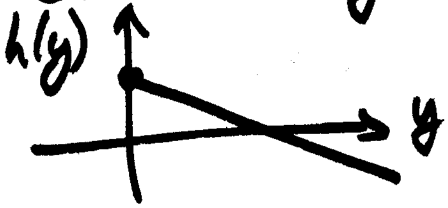


② Logistic growth

$$y' = h(y)\,y$$

$h(y)$: 1) $h(y) \approx r$ for small y
2) $h(y) < 0$ when y large
3) $h(y) \searrow 0$ as $y \nearrow$

Choose $h(y) = r - ay$
$a > 0$



$$\Rightarrow \quad y' = (r - ay)\,y = r\left(1 - \frac{a}{r}y\right)y$$

$$K = \frac{r}{a} \Rightarrow \boxed{y' = r\left(1 - \frac{y}{K}\right)y}$$

Solution to logistic growth problem:

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

$$\int \frac{dy}{\left(1 - \frac{y}{K}\right)y} = \int r\,dt \Rightarrow \int \left(\frac{1/K}{1 - \frac{y}{K}} + \frac{1}{y}\right)dy = \int r\,dt$$

partial fractions

$$\ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + C$$

$$\ln\left|\frac{y}{1 - \frac{y}{K}}\right| = rt + C$$

$$\frac{y}{1 - \frac{y}{K}} = Ce^{rt} \Rightarrow \boxed{y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}}$$

$y(0) = y_0$

$\downarrow$

$C = K - y_0$