Math 214.002

Lecture 8.

Ex. 1 $\quad tx' = x + 3t^2 \quad , \quad x(1) = 2 \quad \leftarrow t_0 = 1$

is there a solution to this IVP and is it unique? specify where solution is valid.

1) linear problem $\Rightarrow$ Thm 1 is tested

2) $p(t) = -\frac{1}{t}$ $\qquad$ $tx' - x = 3t^2$

$g(t) = 3t$ $\qquad$ $x' - \frac{1}{t}x = 3t$

$t = 0 \leftarrow$ pt. of discontinuity for $p(t)$.

Thm 1 conditions hold in $t < 0$ or $t > 0$ regions.

$t_0 = 1 > 0 \Rightarrow$ unique solution exists in $t > 0$.

Ex. 2. $\quad x' = (x-1)\cos(xt) \; , \; x(0) = 1$

Question: can we find $\varphi(t)$ which solves this problem without integrating.

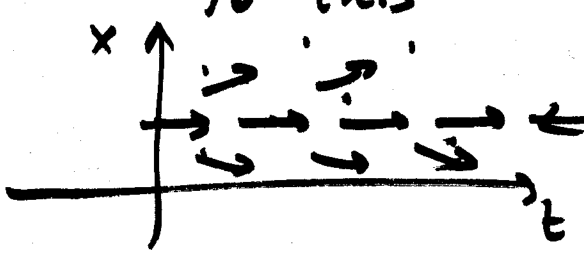
1) nonlinear $\Rightarrow$ Thm 2 is tested.

2) $f(x,t) = (x-1)\cos(xt)$

$$\frac{\partial f(x,t)}{\partial x} = \cos(xt) - (x-1)\cdot t \cdot \sin(xt)$$

continuous for all t & x.

$\Rightarrow$ by Thm 2 we have unique solution to this IVP (in fact, any IVP $x(t_0) = y_0$).



constant solution is where $x' = 0$.

$x' = (x-1)\cos(xt)$

$x \equiv 1$

$\boxed{\varphi(t) \equiv 1}$ is const. sol.

Ex. 3 $\quad y' = y^{1/3} \quad y(0) = \underline{0}.$

nonlinear problem $\Rightarrow$ check Thm 2.

$f = y^{1/3} \qquad \frac{\partial f}{\partial y} = \frac{1}{3} y^{-2/3}$ $\leftarrow$ discontinuous at $\underline{y=0}$.

Continuous for all y $\Rightarrow$ Thm 2 gives existence

For this IVP, uniqueness part of Thm 2 does not apply: you could have more than one solution passing through $(0,0)$.

$$\frac{dy}{dt} = y^{1/3} \Rightarrow \int \frac{dy}{y^{1/3}} = \int dt$$

$$\int y^{-1/3}\, dy$$

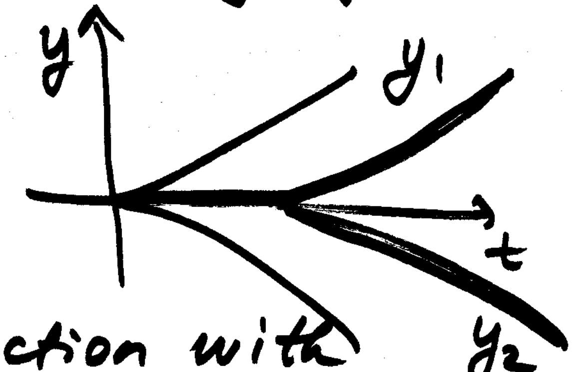
$$y^{2/3} = \frac{2}{3}(t+C) \Leftarrow \frac{3}{2} y^{2/3} = t + C \Rightarrow C = 0 \text{ from } y(0)=0.$$

$$\boxed{y = \left(\frac{2}{3}(t+C)\right)^{3/2}}$$

$y_1 = \left(\frac{2}{3}t\right)^{3/2}$ - solution to IVP. $t \geq 0$.

$y_2 = -\left(\frac{2}{3}t\right)^{3/2}$ — another solution to $y' = y^{1/3}$

$$y = \begin{cases} 0, & 0 \leq t < t_0 \\ \pm\left(\frac{2}{3}(t-t_0)\right)^{3/2}, & t \geq t_0 \end{cases}$$



Solution to this IVP is not unique, no contradiction with Thm 2.

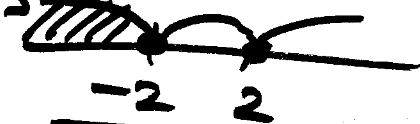
Linear	Nonlinear
1) discontinuities of $y(t)$ happen only at discont. of $p(t)$ or $g(t)$	2) solution can be disc. at places where f is continuous.

Linear	Nonlinear
2) Always has <u>general solution</u> with one arbitrary constant that gives all solutions to the equation	2) Usually "general solution" can not be found

<u>Examples.</u> Determine where solution exists:

(a) $(4-t^2)y' + 2ty = 3t^2$, $y(-3) = 1$ $\rightarrow$ $\dfrac{2t}{4-t^2} = \dfrac{2t}{(2-t)(2+t)}$

(b) $(\ln t)y' + y = \cot(t)$, $y(2) = 3$



$\boxed{t < -2}$

<u>§2.5 Autonomous equations.</u>

$\left[\dfrac{dy}{dt} = f(y)\right]$ ← no time dependence in the r.h.s.

always separable equations

<u>Ex.</u> $\dfrac{dy}{dt} = y^2(3-2y)(5+y)^3$ $\quad \lim\limits_{t\to\infty} y(t) = ?$

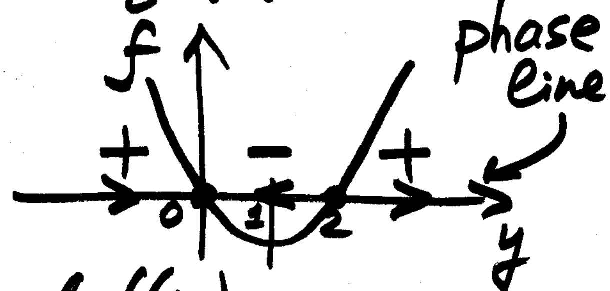
Qualitative analysis of autonomous equations can be done by analyzing the right-hand side: $\quad \dfrac{dy}{dt} = f(y)$

Things to compute: 1) where $f(y) \geq 0$ or $f(y) < 0$
2) where $f'(y) \geq 0$ or $f'(y) \leq 0$.

Example. $\frac{dy}{dt} = 2y(y-2)$ $\quad \lim_{t\to\infty} y(t) = ?$

1) Graph $f(y)$

$f(y) = 2y(y-2)$



2) Find zeros $\quad \boxed{\begin{matrix} y=0 \\ y=2 \end{matrix}}$ > zeros of $f(y)$ (equilibrium solutions of DE)

2) Find $f'(y)=0$ critical points

$f' = 4y - 4 = 0$ (inflection points in DE).

$\boxed{y=1}$ $\quad \frac{dy}{dt} = f(y) \Rightarrow y'' = f \cdot f'$

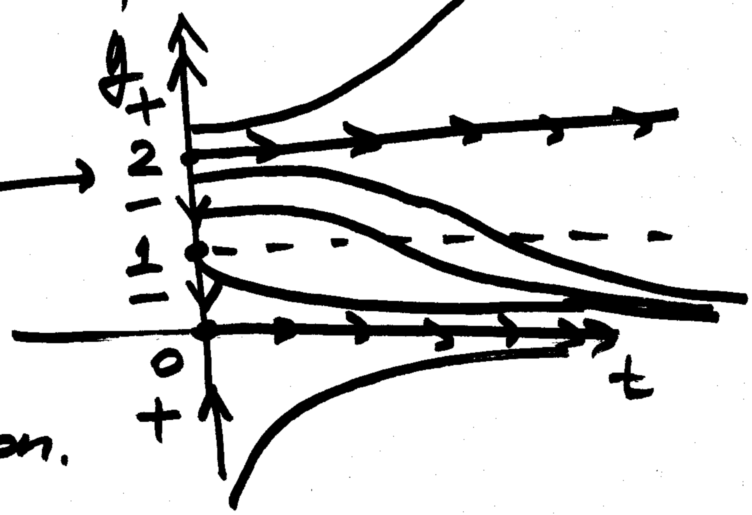
3) Determine intervals of $f>0$, $f'>0$

	f	f'	$f \cdot f'$
$y<0$	+	−	−
$0<y<1$	−	−	+
$1<y<2$	−	+	−
$y>2$	+	+	+

concave down

concave up

4) Phase portrait:



$y=2$ unstable solution

$y=0$ asymptotically stable solution.