Math 214.002

Lecture 4.

§2.1 Integrating factor.

$\frac{dy}{dt} + p(t)y = g(t)$ ← Standard form

Part 1. Reduce to Standard form

Ex. 1) $y' + 2y = 3$

2) $y' + \underline{2t}y = \underline{3e^t}$ $\quad p(t) = 2t,\ g(t) = 3e^t$

3) $5y' = y - 2$

$\rightarrow 5y' - y = -2$

$y' - \underline{\tfrac{1}{5}}y = \underline{-\tfrac{2}{5}}$ $\quad p(t) = -\tfrac{1}{5},\ g(t) = -\tfrac{2}{5}$

4) $ty' = y + \cos t$

$ty' - y = \cos t \Rightarrow y' - (\tfrac{1}{t})y = \frac{\cos t}{t}$

$p(t) = -\frac{1}{t},\ g(t) = \frac{\cos t}{t}$

Part 2. Find integrating factor.

$[\mu(t) = e^{\int p(t)\,dt}] \leftarrow$ Solves $\mu' = p(t)\mu$

Ex. 1) if $y' + 2y = 3 \Rightarrow \mu(t) = e^{\int 2\,dt} = e^{2t+C} = \underbrace{e^C}_{C} \cdot e^{2t} = \underline{Ce^{2t}}$

$p(t) = 2$ $\quad \boxed{\mu(t) = e^{2t}}$ $\quad$ pick $C = 1$

2) $y' + \underline{(\tfrac{1}{t})}y = e^t$ $\quad \mu(t) = e^{\int \frac{1}{t}dt} = e^{\ln|t|} = t$

$p(t) = \frac{1}{t}$ $\quad \boxed{\mu(t) = t}$

Part 3) If μ was found, such that $\mu' = p(t)\mu$

$$\mu = e^{\int p(t)\,dt}$$

Original equation:

$$\Rightarrow \quad y' + p(t)y = g(t) \quad \Big| * \mu(t)$$

transforms to

$$[\mu(t)y(t)]' = \mu(t)g(t)$$

$$\mu(t)y(t) = \int \mu(t)g(t)\,dt + C$$

$$y(t) = \frac{1}{\mu(t)} \cdot \left[\int \mu(t)g(t)\,dt + C\right]$$

Ex. 1) $y' + 2y = 3 \qquad \mu(t) = e^{2t}$

$$e^{2t}y' + 2e^{2t}y = 3e^{2t}$$

$$[e^{2t}y(t)]' = e^{2t}y' + 2e^{2t}y$$

$$[e^{2t}y(t)]' = 3e^{2t}$$

Integrate wrt t : $e^{2t}y(t) = \left[\int 3e^{2t}\,dt\right] + C$

$$e^{2t}y(t) = \frac{3}{2}e^{2t} + C$$

$$\boxed{y(t) = \frac{3}{2} + C \cdot e^{-2t}}$$

2) $ty' + 2y = t^2$

$\boxed{y' + \left(\frac{2}{t}\right)y = t} \qquad p(t) = \frac{2}{t}, \; g(t) = t$

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{2}{t}\,dt} = e^{2\ln t} = \left(e^{\ln t}\right)^2 = \underline{\underline{t^2}}$$

$$\mu(t) = t^2$$

$$(t^2 y)' = t^2 \cdot t \quad \leftarrow$$

Reminder:

$$y' + \left(\frac{2}{t}\right) y = t$$

$$\underbrace{\mu y' + \left(\frac{2}{t}\right)\mu y}_{(\mu y)'} = \mu t$$

$$(t^2 y)' = t^3$$

$$t^2 y = \frac{t^4}{4} + C$$

$$\boxed{y(t) = \frac{t^2}{4} + C t^{-2}}$$

3) $$\underbrace{t y' + 3y = 2t^3 + 1}$$ $\mu(t) = t^3$

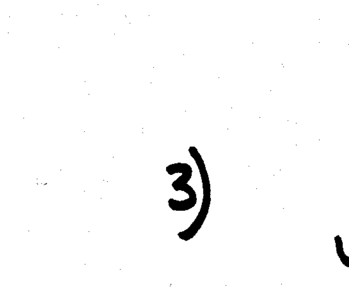
$$y' + \frac{3}{t} y = 2t^2 + \frac{1}{t}$$

$$\longrightarrow p(t) = \frac{3}{t} \Rightarrow \mu(t) = e^{\int \frac{3}{t} dt} = t^3$$

$$t^3\left(y' + \frac{3}{t} y\right) = 2t^5 + t^2$$

$$\longrightarrow (t^3 y)' = 2t^5 + t^2$$

$$t^3 y = 2 \cdot \frac{t^6}{6} + \frac{t^3}{3} + C$$



$$\boxed{y = \frac{1}{3} \cdot t^3 + \frac{1}{3} + C t^{-3}}$$

solution has a discontinuity at $t=0$.

<u>Initial value problem</u>:

$y(1) = 1$

pick a branch for $t>0$.

$$y(1) = \frac{1}{3} + \frac{1}{3} + C = 1 \Rightarrow C = \frac{1}{3}$$

Solution to IVP: $y(t) = \frac{1}{3}\left(1 + t^3 + t^{-3}\right)$, $t>0$

Two techniques for linear ODE 1st order:

Example

1) $y' + ay = b$

By Integrating factor

$\mu = e^{at}$

$(e^{at}y)' = b e^{at}$

$e^{at} y = \frac{b}{a} e^{at} + C$

$y(t) = \frac{b}{a} + Ce^{-at}$

By separation of variables

$\frac{dy}{dt} = b - ay$

$\frac{dy}{dt} = -a\left(y - \frac{b}{a}\right)$

$\frac{dy}{y - \frac{b}{a}} = -a\,dt$

$\ln\left|y - \frac{b}{a}\right| = -at + C$

$y - \frac{b}{a} = Ce^{-at}$

$y = \frac{b}{a} + Ce^{-at}$