Math 214.002
Lecture 24.

Final: 4:30 - 7:15pm Tue Dec 9 same place

1 formula sheet

10 questions, 100 pts

partial credit

(1) §2. 1st order equations.

1) Integrating factor

$$y' + p(t)y = g(t) \text{ — standard form}$$

$$ty' + y = e^t \qquad \mu(t) = e^{\int p(t)dt}$$

$$y' + \frac{1}{t}y = \frac{e^t}{t} \qquad \mu(t) = e^{\int \frac{1}{t}dt} = e^{\ln|t|} = t$$

$$p(t) = \frac{1}{t} \qquad g(t)$$

multiply both parts by $\mu(t)$, use product rule:

$$(\mu(t)y(t))' = \mu(t)g(t)$$

$$(ty)' = e^t \Rightarrow ty(t) = e^t + C$$

$$y(t) = \frac{e^t}{t} + \frac{C}{t}$$

2) Separable equations:

$$\frac{dy}{dt} = \frac{y+1}{t^2} \Rightarrow \int \frac{dy}{y+1} = \int \frac{dt}{t^2}$$

$$\ln|y+1| = -\frac{1}{t} + C$$

a) if $y(2) = y_0 \Rightarrow$ find C value

b) if you're asked for explicit form of solution: $y(t) = \underbrace{\ldots}_{\text{function of } t \text{ only}}$ (rhs has no y in it)

Pick the right branch!

3) Modeling — mixture/falling body.

<u>Mixture</u>: $\frac{dA}{dt} = C_{in} r_{in} - C_{out} r_{out}$

$$V(t) = V_0 + (r_{\cancel{out}\ in} - r_{\cancel{in}\ out})t$$

well-mixed solution means: $C_{out} = C_{\text{inside the tank}}$

$$C_{out} = \frac{A(t)}{V(t)}$$

Suppose your solution is $A(t) = 100 + 100e^{-t}$ (salt)

$V(t) = 200 \leftarrow$ total volume

lim. contentration: $\lim_{t\to\infty} \frac{A(t)}{V(t)} = \frac{100}{200} = \frac{1}{2}$

Falling body:

v $\quad mv' = mg - kv$

<u>OR</u> $mv' = mg + kv$

② <u>2nd order linear equations</u>.

1) $ay'' + by' + cy = 0 \leftarrow$ know how to solve these for all a, b, c

General solution looks like $\underline{\underline{c_1}}\, y_1(t) + \underline{\underline{c_2}}\, y_2(t)$

2) Non-homogeneous equations:

$$ay'' + by' + cy = g(t)$$

General solution: $y(t) = \underbrace{c_1 y_1(t) + c_2 y_2(t)}_{\text{gen sol. for hom. eqn.}} + \underbrace{y_p(t)}_{\text{particular solution to non. hom. eqn}}$

Review <u>undetermined coefficients</u>.

3) Independance & Wronskian

$$W(y_1,y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

$$W(y_1,y_2)(t) = Ce^{-\int p(t)dt}$$ - Abel formula

Fund. set : $y(t) = C_1 y_1(t) + C_2 y_2(t)$ gen. sol.

equivalent $\left\{ \begin{array}{l} \text{lin. indep. functions} \\ \text{i.e. } W(y_1,y_2) \neq 0 \\ \Rightarrow \{y_1, y_2\} \text{ - fund. set of solutions} \end{array} \right.$

(3) Laplace transform.

1). Inverse/direct Laplace transform

→ partial fractions

$$F(s) = \frac{2}{(s^2+1)(s-1)}$$

→ completing the square

$$F(s) = \frac{3}{s^2+6s+10} = \frac{3}{(s+3)^2+1}$$

2). $y'' + 2y' - 3y = t + \delta(t)$, $y(0)=0$, $y'(0)=0$.

$y'' \to s^2F(s) - sy(0) - y'(0)$

3) Shifts with Laplace

$$F(s) = \frac{e^{-3s}}{s+2} \xrightarrow{\mathcal{L}^{-1}} ?$$

$$f(t) = \begin{cases} 1, & 0<t<2 \\ t, & t \geq 2 \end{cases} \xrightarrow{\mathcal{L}} ?$$

$$f(t) = 1 + (t-1)\, u_2(t)$$

$$g(t) = 1 + (t-1)u_2(t) = 1 + (\tau+1)u_2(t)$$

$$\downarrow \qquad \tau = t-2$$

$$f(t-2) \qquad t-1 = \tau+2-1 = \tau+1$$

$$f(\tau) = \tau + 1$$

$$f(t-c)u_c(t) =$$

$$f(t) = t+1 \qquad c = 2$$

$$\mathcal{L}\{g\} = \frac{1}{s} + e^{-2s} \cdot \mathcal{L}\{t+1\} = \frac{1}{s} + e^{-2s} \cdot \left(\frac{2}{s^2} + \frac{1}{s}\right)$$