Math 217.002

Lecture 23.

§6.3–6.5

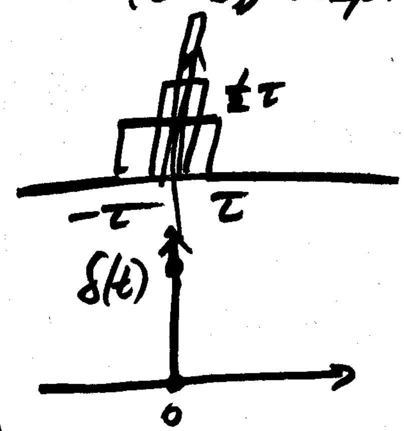
Discontinuous forcing.

$$mu'' + \gamma u' + ku = F(t)$$

if $F(t)$ acts half of the time, or only acts once, $F(t)$ will be represented by step functions ($u_c(t)$) or delta-functions ($\delta(t-c)$) resp.

Delta function $\equiv$ impulse function

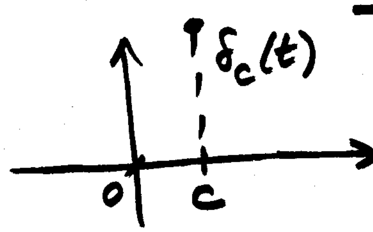
$$d_\tau(t) = \begin{cases} \frac{1}{2}\tau, & -\tau < t < \tau \\ 0, & \text{otherwise} \end{cases}$$



$$\lim_{\tau \to 0} d_\tau(t) = 0, \quad t \neq 0.$$

$$\delta(t) = 0, \quad t \neq 0, \quad \int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$

$$\delta_c(t) = 0, \quad t \neq c$$



Def. $\delta_c(t) = \delta(t-c)$

Summary.

1) Def. of $u_c(t) = \begin{cases} 1, & t \geq c \\ 0, & t < c \end{cases}$

2) $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} F(s), \quad F(s) = \mathcal{L}\{f\}$

3) $\mathcal{L}\{e^{ct} f(t)\} = F(s-c), \quad F(s) = \mathcal{L}\{f\}.$

4) $\mathcal{L}\{\delta(t-c)\} = e^{-cs}$

Ex. 1 Rewrite function in terms of step functions.

$$f(t) = \begin{cases} t, & t<2 \\ (t-2)^2, & t \geq 2 \end{cases}$$

$$f(t) = (\underset{\substack{\uparrow \\ t<2}}{t}) + ((t-2)^2 - t)u_2(t)$$

Take Laplace transform:

$$\mathcal{L}\{f\} = \frac{1}{s^2} + e^{-2s}\mathcal{L}\{t^2 - t - 2\} = \boxed{\frac{1}{s^2} + e^{-2s}\left(\frac{2}{s^3} - \frac{1}{s^2} - \frac{2}{s}\right)}$$

$(t-2)^2 - t = f(t-2)$ In order to apply $\left[\begin{array}{l} f(t-2)u_2(t) \longrightarrow e^{-2s}F(s) \\ F = \mathcal{L}\{f\} \end{array}\right]$

$((t-2)^2 - (t-2) - 2$

Method 2: $\tau = t-2$

$t = \tau + 2$

$\tau^2 - \tau - 2 = f(\tau)$

Ex. 2 $f(t) = \begin{cases} t, & t<1 \\ \boxed{e^t + e^{-t}}, & 1 \leq t < 2 \\ 1, & 2 \leq t \end{cases}$

$$f(t) = (t) + (e^t + e^{-t} - t)u_1(t) + (1 - e^t - e^{-t})u_2(t)$$

$$\underbrace{(e^t + e^{-t} - t)}_{f(t-1)}u_1(t) = f(t-1)u_1(t)$$

$\tau = t-1$

$t = \tau + 1$

$$f(\tau) = e^{\tau+1} + e^{-(\tau+1)} - (\tau+1)$$

$e^{\tau+1} = e \cdot e^{\tau}$

$e\mathcal{L}(e^{\tau})$

Ex. 3. $y'' + 2y' + 10y = 10 + \delta(t)$, $y(0) = 1$, $y'(0) = 0$.

$$s^2F(s) - s\underset{\substack{\| \\ 1}}{y(0)} - \cancel{y'(0)}^{\,0} + 2\Big(sF(s) - \underset{\substack{\| \\ 1}}{y(0)}\Big) + 10F(s) = \frac{10}{s} + \underset{\substack{\uparrow \\ 1}}{e^{-0s}}$$

$$(s^2 + 2s + 10)F(s) - s - 2 = \frac{10}{s} + 1$$

$$F(s) = \frac{s+2}{s^2+2s+10} + \frac{10}{s(s^2+2s+10)} + \frac{1}{s^2+2s+10}$$

$$\underbrace{s^2+2s}+10 = (s^2+2s+1)+9 = (s+1)^2+9$$
$$= (s+1)^2 + 3^2$$

$$F(s) = \frac{s+3}{(s+1)^2+3^2} + \frac{10}{s(s^2+2s+10)} =$$

$$\frac{10}{s(s^2+2s+10)} = \frac{a}{s} + \frac{bs+c}{s^2+2s+10} = \frac{1}{s} - \frac{s+2}{s^2+2s+10}$$

$$\underline{as^2} + \underline{2as} + 10a + \underline{bs^2} + \underline{cs} = 10$$

$$\begin{cases} a + b = 0 & b = -a = -1 \\ 2a + c = 0 & c = -2a = -2 \\ 10a = 10 \Rightarrow & a = 1 \end{cases}$$

$$= \frac{1}{(s+1)^2+3^2} + \frac{1}{s}$$

$$\mathcal{L}^{-1}(F(s)) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{3}{(s+1)^2+3^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s}\right)$$

$$\boxed{y(t) = \frac{1}{3}e^{-t}\sin 3t + 1}$$

$$\mathcal{L}(e^{at}\sin bt) = \frac{b}{(s-a)^2+b^2}$$

Ex. 4. $y^{(4)} - y = 0$, $y(0) = 1$, $y'(0) = 1$, $y''(0) = 1$, $y'''(0) = 0$.

Laplace of both sides:

$$s^4F(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - F(s) = 0.$$

$$(s^4 - 1)F(s) - s^3 - s^2 - s = 0$$

$$F(s) = \frac{\boxed{s^3+s^2+s}}{s^4-1} = \frac{a}{s-1} + \frac{b}{s+1} + \frac{cs+d}{s^2+1}$$

$$s^4-1 = (s^2-1)(s^2+1) = (s-1)(s+1)(s^2+1)$$

$$a(s+1)(s^2+1) + b(s-1)(s^2+1) + (cs+d)(s^2-1) = s^3+s^2+s$$

<u>Ex. 5</u>. (a) $F(s) = e^{-3s} \cdot \frac{1}{s+2}$

(b) $F(s) = 4\,\frac{e^{-2s}}{s^2+4}$