Math 214.002
Lecture 22.

Ex. $F(s) = \dfrac{s^2+9}{9s-s^3}$ Find $\mathcal{L}^{-1}(F(s))$

$$\frac{s^2+9}{9s-s^3} = \cancel{s^2+9}\ \frac{s^2+9}{s(9-s^2)} = \frac{s^2+9}{-s(s-3)(s+3)}$$

$$\frac{-s^2-9}{s(s-3)(s+3)} = \frac{a}{s} + \frac{b}{s-3} + \frac{c}{s+3}$$

$$a(s^2-9) + bs(s+3) + cs(s-3) = -s^2-9$$

$$\underline{as^2} - 9a + \underline{bs^2} + 3bs + \underline{cs^2} - 3cs = -\underline{s^2} - 9$$

$$\begin{cases} a+b+c=-1 \\ 3b-3c=0 \\ -9a=-9 \end{cases} \Rightarrow \underline{a=1}$$

$$b+c=-2$$
$$3b-3c=0 \qquad b=c$$
$$\underline{b=c=-1}$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{s-3} - \frac{1}{s+3}\right) = 1 - e^{3t} - e^{-3t}$$

Ex. $F(s) = \dfrac{se^{-2s} - e^{-5s}}{s^2+6s+10} = \dfrac{s}{s^2+6s+10} \cdot e^{-2s} - \dfrac{e^{-5s}}{s^2+6s+10}$

$$s^2+6s+10 = (s^2+6s+9)+1 = (s+3)^2+1$$

$$\frac{s}{s^2+6s+10} = \frac{(s+3)-3}{(s+3)^2+1} = \frac{s+3}{(s+3)^2+1} - \frac{3}{(s+3)^2+1}$$

$$\frac{s-a}{(s-a)^2+b^2} \qquad\qquad \frac{b}{(s-a)^2+b^2}$$

$$F(s) = e^{-2s}\cdot \underbrace{\frac{s+3}{(s+3)^2+1}}_{\mathcal{L}(e^{-3t}\cos t)} - 3\cdot e^{-2s}\underbrace{\frac{1}{(s+3)^2+1}}_{\mathcal{L}(e^{-3t}\sin t)} - e^{-5s}\underbrace{\frac{1}{(s+3)^2+1}}$$

By line 13, $\boxed{\mathcal{L}^{-1}(e^{-cs}F(s)) = u_c(t)f(t-c)}$

$$F(s) = \mathcal{L}\{f\}$$

$$e^{-2s}\cdot \underbrace{\mathcal{L}(e^{-3t}\cos t)}_{F(s)} \xrightarrow{\mathcal{L}^{-1}} u_2(t)f(t-2)$$

$$f(t) = e^{-3t}\cos t$$

$$\Rightarrow \mathcal{L}^{-1}\left(e^{-2s}\cdot\frac{s+3}{(s+3)^2+1}\right) = u_2(t)e^{-3(t-2)}\cos(t-2)$$

$$\mathcal{L}^{-1}\left(e^{-2s}\cdot\frac{1}{(s+3)^2+1}\right) = u_2(t)e^{-3(t-2)}\sin(t-2)$$

$$\mathcal{L}^{-1}\left(e^{-5s}\cdot\frac{1}{(s+3)^2+1}\right) = u_5(t)e^{-3(t-5)}\sin(t-5)$$

<u>Answer:</u> $u_2(t)e^{-3(t-2)}\cos(t-2) - 3u_2(t)e^{-3(t-2)}\sin(t-2)$
$- u_5(t)e^{-3(t-5)}\sin(t-5)$

$$= u_2(t)f_1(t-2) - 3u_2(t)f_2(t-2) - u_5(t)f_3(t-5)$$

where $f_1 = e^{-3t}\cos t$
$f_2 = f_3 = e^{-3t}\sin t$

<u>Ex.</u> $F(s) = \dfrac{1-e^{-3s}}{s^3}$ Find inverse Laplace transform.

$$F(s) = \frac{1}{s^3} - e^{-3s}\cdot\frac{1}{s^3}$$

$\mathcal{L}(t^n) = \dfrac{n!}{s^{n+1}}$ $\quad n=2 \quad \mathcal{L}^{-1}\left(\dfrac{2}{s^3}\right) = t^2$

$$\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) = \frac{1}{2}\cdot t^2 = f(t)$$

$$F(s) = \frac{1}{s^3} - e^{-3s}\cdot\frac{1}{s^3}$$

$$\mathcal{L}^{-1}(F(s)) = f(t) - u_3(t)f(t-3)$$

$$\left[\mathcal{L}^{-1}\left(e^{-cs}F(s)\right) = \underline{u_c(t)f(t-c)}\right]$$

<u>Ex.</u> $\mathcal{L}^{-1}\left(e^{-2s}\cdot\frac{3!}{s^4}\right) = u_2(t)(t-2)^3$

<u>Ex.</u> $\mathcal{L}^{-1}\left(\frac{2e^{-3s}}{s^2-1}\right) = \mathcal{L}^{-1}\left(\frac{e^{-3s}}{s-1} - \frac{e^{-3s}}{s+1}\right) =$

$$\frac{2}{s^2-1} = \frac{a}{s-1} + \frac{b}{s+1} = \frac{1}{s-1} - \frac{1}{s+1}$$

$$as + a + bs - b = 2$$

$a + b = 0 \qquad a = -b$

$a - b = 2 \qquad 2a = 2 \quad a = 1 \quad b = -1$

$= u_3(t)f_1(t-3) - u_3(t)f_2(t-3)$

$f_1(t) = e^t$

$f_2(t) = e^{-t}$

<u>Writing discontinuous functions as step functions:</u>

$$f = u_1(t) - t u_2(t) = \begin{cases} 1-t, & t \geq 2 \\ 1, & 1 \leq t < 2 \\ 0, & t < 1 \end{cases}$$

Opposite direction: $g(t) = \begin{cases} t, & t \geq 2 \\ 1, & t < 2 \end{cases}$

$$\boxed{g(t) = 1 + (t-1)u_2(t)}$$

If $g(t) = \begin{cases} t^2, & t \geq 3 \\ t, & 1 \leq t < 3 \\ 2, & t < 1 \end{cases}$

$$g(t) = \underset{\substack{\uparrow \\ t<1}}{2} + (\underset{\substack{\uparrow \\ 1\le t<3}}{t-2})\,u_1(t) + (t^2-t)\,u_3(t)$$

$$t<1 \Rightarrow 2 \ \checkmark$$

$$1 \le t < 3 \Rightarrow 2 + (t-2) = t \ \checkmark$$

$$t \ge 3 \Rightarrow 2 + (t-2) + t^2 - t = t^2 \ \checkmark$$