Math 214.002

Lecture 21.

Midterm: 8 questions, no multiple choice

(1) Solutions of 2nd order eqns with const coefficients

$$ay'' + by' + cy = 0$$
$$ar^2 + br + c = 0$$

r_1, r_2 - roots

1) $r_1 \neq r_2$ real

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

2) $r_{1,2} = \lambda \pm i\mu$

$$y = C_1 e^{\lambda t}\cos\mu t + C_2 e^{\lambda t}\sin\mu t$$

3) $r_1 = r_2$ real

$$y = C_1 e^{rt} + C_2 t e^{rt}$$

(2) Independence & Wronskian

→ Def of Wronskian

→ Abel's formula

Ex. $y_1 = t$, $y_2 = t^2$ - solutions to some DE.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2 \neq 0.$$

$t \neq 0$

$\Rightarrow$ 1) y_1, y_2 - lin. indep.

2) $\{y_1, y_2\}$ - fund. set

3) $y = C_1 y_1 + C_2 y_2 = C_1 t + C_2 t^2$ ← gen. sol.

③ Reduction of order / Abel formula for finding 2nd lin. indep. solution.

$$t^2y'' - 4ty' + 6y = 0, \quad y_1 = t^2$$

$y_2 = v \cdot y_1$

$\cancel{y_1 v'' + p y_1 y'}$

$$y'' - \frac{4}{t}y' + \frac{6}{t^2}y = 0 \qquad \boxed{y_1 v'' + (2y_1' + p y_1)v' = 0}$$

$$p = -\frac{4}{t} \quad y_1 = t^2$$

$u = v'$
$u' = v''$

$$t^2 v'' + (4t - \frac{4}{t} \cdot t^2)v' = 0$$

$$t^2 v'' = 0$$

$$t^2 u' + (4t - 4t)u = 0 \Rightarrow t^2 u' = 0.$$

$$u' = 0 \quad u = c$$

$$v' = c$$

$$v = ct + c_1$$

$$\Rightarrow y_2 = (ct + c_1)\underset{y_1 = t^2}{y_1} = ct^3 + \boxed{c_1 t^2} \sim t^3$$

$\cancel{y_2 = t^2}$

indep. part from $y_1 = t^2$

$$y = C_1 y_1 + C_2 y_2$$
$$= C_1 t^2 + C_2 t^3 \leftarrow \text{gen. sol.}$$

④ Undetermined coefficients.

1) Find form of y_p
2) Find gen. sol. of nonhom. equation.

Ex. $y'' + 2y' - 3y = \boxed{5e^{3t}} + \cos 2t$

$y'' + 2y' - 3y = 0.$ $\quad (r+3)(r-1) = 0$

$r = -3,\ r = +1$

$y_{hom.} = C_1 e^{-3t} + C_2 e^{+t}$

$y_p = a e^{\boxed{3}t} + b \cos 2t + c \sin 2t$

Ex. $y'' + 2y' - 3y = 5e^{-3t} \cos 2t$

$y_p = a e^{-3t}(b \cos 2t + c \sin 2t)$

$r = 3,\ r = +1$ $\quad$ no duplication

if $r = -3 \pm 2i$ was a root of char. eqn. $\Rightarrow$ would need an extra (t).

if $r = -3 \pm 2i$

$y_{hom} = C_1 e^{-3t} \cos 2t + C_2 e^{-3t} \sin 2t$

if $r = \pm 2i$

$y_{hom.} = C_1 \cos 2t + C_2 \sin 2t$

$y_p = a e^{-3t} \cos 2t + b e^{-3t} \sin 2t$

if $r_1 = 2,\ r_2 = 2$

$y_{hom.} = C_1 e^{2t} + C_2 t e^{2t}$

(5) Mechanical vibrations.

metric system: $\quad F = ma$

mass : kg

force : Newtons $= \frac{kg \cdot m}{s^2}$, length $= [m]$

→ unforced / forced

→ damped / undamped

Model vibrating system: $mu'' + \gamma u' + ku = F(t)$

#2 Quiz 9:

$F_d = 8N = \gamma |u'|$

$F_d = 8N$

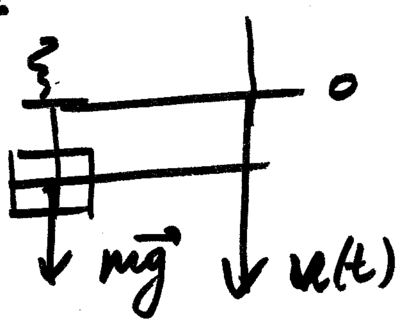
↑ velocity $= 2 \frac{m}{sec}$

$\boxed{\gamma = 4 \frac{kg}{sec}}$

$\left.\begin{array}{l} m = 2\,kg \\ L = 1\,m \end{array}\right\} \Rightarrow \boxed{k = \frac{mg}{L}} = \frac{20}{1} = 20$

$u(0) = 1$

$u'(0) = 3\sqrt{3} - 1$



Key points:

1) natural frequency $\quad mu'' + ku = 0$

$u = R\cos(\omega t - \delta)$

$T = \frac{2\pi}{\omega}$ – period, amplitude, nat. frequency

2) $mu'' + ku = F(t)$

$F(t) = \cos \alpha t$

Resonance occurs when: $\boxed{\alpha = \omega}$

↑ nat. frequency

3) Damped case:

$u = Re^{-\gamma t/2m} \cos(\mu t - \delta)$

↑ quasifrequency

$F = 10\cos 2t \qquad k = 12\,\frac{N}{m}$

$\Rightarrow \quad mu'' + ku = F(t)$

$mu'' + 12u = 10\cos 2t$

$mu'' + 12u = 0$

$u'' + \frac{12}{m}u = 0 \qquad r^2 + \frac{12}{m} = 0$

$r = \pm i\sqrt{\frac{12}{m}} = \pm i\,\frac{2\sqrt{3}}{\sqrt{m}}$

$u = C_1 \cos \underbrace{\sqrt{\frac{12}{m}}}\, t + C_2 \sin \sqrt{\frac{12}{m}}\, t$

Resonance: $w = 2$

$\frac{2\sqrt{3}}{\sqrt{m}} = 2 \Rightarrow m = 3$

⑥ <u>Laplace transform</u>.

1) Take direct, inverse Laplace transform
2) Solve IVP with Laplace

<u>Ex. 1</u> $F(s) = \frac{4}{s^2+4s+5}$

$F(s) = \frac{b}{(s-a)^2+b^2} \xrightarrow{\mathcal{L}^{-1}} e^{at}\sin bt$ (line 9)

$s^2+4s+5 = \underline{\underline{(s+2)^2}}+1$

$F(s) = \frac{4}{(s+2)^2+1} \qquad a=-2 \quad b=1$

$= 4\cdot\frac{1}{(s+2)^2+1} \xrightarrow{\mathcal{L}^{-1}} \boxed{4\cdot e^{-2t}\sin t}$

⑦ <u>Step functions</u>. $\boxed{u_c(t) = \begin{cases} 1, & t \geq c \\ 0, & t < c \end{cases}}$