Math 214.002

Lecture 20.

Ex. $y'' - y' - 6y = 2$, $y(0)=0$, $y'(0)=0$

Laplace of both sides:

$\mathcal{L}\{y'' - y' - 6y\} = \mathcal{L}\{2\}$ $\quad 2\cdot\mathcal{L}\{1\} = \frac{2}{s}$ by Line 1

$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\underbrace{\mathcal{L}\{y\}}_{F(s)} = \mathcal{L}\{2\}$ ⊛

$F(s)$ ← notation

Line 18: $\mathcal{L}\{y''\} = s^2F(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} = s^2F(s)$ ← $n=2$

$\mathcal{L}\{y'\} = sF(s) - \cancel{y(0)}^{0} = sF(s)$ ← $n=1$

$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$

Back to ⊛:

$$s^2F(s) - sF(s) - 6F(s) = \frac{2}{s}$$

$$(s^2 - s - 6)F(s) = \frac{2}{s}$$

$$F(s) = \frac{2}{s(s^2-s-6)}$$ ← Laplace transform of our solution $y(t)$

Take inverse transform:

Simplify $F(s)$ as: $F(s) = \dfrac{2}{s(s+2)(s-3)}$

Partial fractions: $\dfrac{a}{s} + \dfrac{b}{s+2} + \dfrac{c}{s-3}$

numerator: $a(s+2)(s-3) + bs(s-3) + cs(s+2) = 2$

$as^2 - as - 6a + bs^2 - 3bs + cs^2 + 2cs = 2$

$$\begin{cases} a + b + c = 0 \\ -a - 3b + 2c = 0 \\ -6a = 2 \end{cases} \qquad 3\times\begin{cases} b + c = \frac{1}{3} \\ -3b + 2c = -\frac{1}{3} \end{cases}$$

$$\underline{a = -\tfrac{1}{3}} \qquad 5c = \cancel{\tfrac{4}{3}}\ \tfrac{2}{3}$$

$$\underline{b = \tfrac{3}{15}}, \qquad \underline{c = \tfrac{2}{15}}$$

$$F(s) = \left(-\frac{1}{3}\right)\frac{1}{s} + \left(\frac{3}{15}\right)\frac{1}{s+2} + \left(\frac{2}{15}\right)\cdot\frac{1}{s-3}$$

$$y(t) = \mathcal{L}^{-1}(F(s)) = \left(-\frac{1}{3}\right)\underbrace{\mathcal{L}^{-1}\left(\frac{1}{s}\right)}_{\substack{\parallel \text{ Line 1} \\ 1}} + \frac{3}{15}\underbrace{\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)}_{\substack{\parallel \text{ Line 2} \\ e^{-2t}}} + \frac{2}{15}\underbrace{\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)}_{\substack{\parallel \\ e^{3t}}}$$

$$\boxed{y(t) = -\frac{1}{3} + \frac{3}{15}e^{-2t} + \frac{2}{15}e^{3t}}$$

$\mathcal{L}^{-1}\left\{\frac{4}{s+5}\right\} = 4e^{-5t}$ Line 2

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \cos 3t$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+9}\right\} = \frac{2}{3}\sin 3t, \qquad \mathcal{L}^{-1}\left\{\frac{5}{s^2+4}\right\} = \frac{5}{2}\sin 2t$$

$$\frac{2}{s^2+9} = \left(\frac{2}{3}\right)\frac{3}{s^2+9} \to \sin 3t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}\cdot t^2$$

$n=2$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = t^2$$

Ex. $y'' + y = \sin 2t$, $y(0)=2$, $y'(0)=1$

$$s^2F(s) - s\underbrace{y(0)}_{2} - \underbrace{y'(0)}_{1} + F(s) = \frac{2}{s^2+4}$$

line 5

$$(s^2+1)F(s) - 2s - 1 = \frac{2}{s^2+4}$$

$$(s^2+1)F(s) = 2s+1+\frac{2}{s^2+4}$$

$$F(s) = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)}$$

$$y = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$

Ex. $[y''] + 4y = e^t$, $\underline{y(0)=0,\ y'(0)=0}$

$$[s^2F(s) - \underbrace{sy(0) - y'(0)}_{0}] + 4F(s) = \underbrace{\frac{1}{s-1}}_{\text{line 2}}$$

line 18

$$(4+s^2)F(s) = \frac{1}{s-1} \Rightarrow F(s) = \frac{1}{(s-1)(s^2+4)}$$

$$= \frac{a}{s-1} + \frac{bs+c}{s^2+4}$$

deg(p)−1 poly

$= p$

$$\underline{as^2} + 4a + \underline{bs^2} - \underline{bs} + \underline{cs} - c = 1$$

$$\begin{cases} a+b=0 \\ -b+c=0 \\ 4a-c=1 \end{cases}$$

$a=-b$

$b=c$

$-4b-b=1$ $\quad b=-\frac{1}{5}$

$a=\frac{1}{5}$ $\quad c=-\frac{1}{5}$

$$F(s) = \frac{1}{5}\cdot\frac{1}{s-1} - \frac{1}{5}\cdot\frac{s}{s^2+4} - \frac{1}{5}\cdot\frac{1}{s^2+4}$$

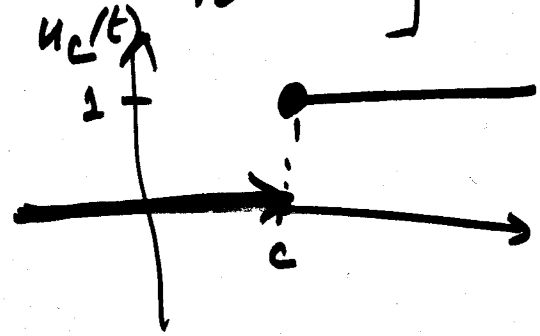
line 2

$$y(t) = \frac{1}{5}\cdot e^t - \frac{1}{5}\cdot\cos 2t - \frac{1}{10}\sin 2t$$

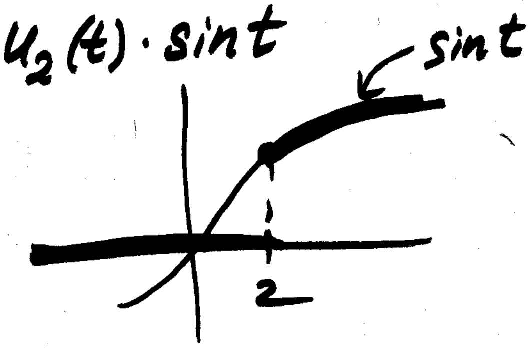
$$\boxed{y(t) = \frac{1}{5}e^{t} - \frac{1}{5}\cos 2t - \frac{1}{10}\sin 2t}$$

Step functions:

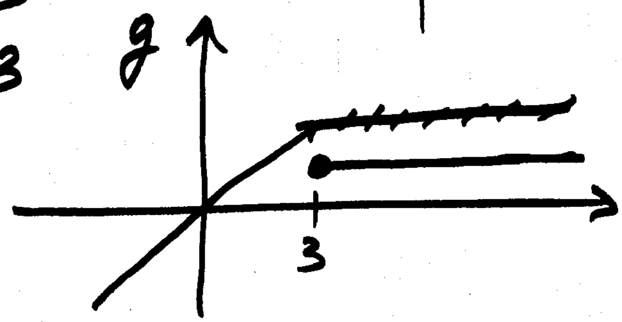
$$u_c(t) = \begin{cases} 1, & t \geq c \\ 0, & t < c \end{cases}$$



$$f(t) = \begin{cases} \sin t, & t \geq 2 \\ 0, & t < 2 \end{cases} = u_2(t) \cdot \sin t$$



$$g(t) = \begin{cases} 2, & t \geq 3 \\ t, & t < 3 \end{cases}$$



Write $g(t)$ as a step function:

$$g(t) = 2 \cdot u_3(t) + t \cdot (1 - u_3(t))$$
$$= t + (2 - t)u_3(t)$$