Math 214.002

Lecture 10

Ex. 1 (Reduction of order)

Given $y_1 = t^3$ is a solution to $t^2y'' + 2ty' - 12y = 0$

Find general solution.

Vibrations: $mu'' + \gamma u' + ku = F(t)$

Special cases:

1) $\gamma = 0 \to$ undamped, $\gamma \neq 0 \to$ damped
2) $F = 0 \to$ free, $F \neq 0 \to$ forced

Things to remember:

1) $mg = kL$ is the condition for getting k.

   L ↑ elongation due to gravity.

2) $g = 32 \frac{ft}{sec^2}$, $1 ft = 12 in$

   English units: ft, lb, sec

   Metric units: m, kg, sec

3) If $u = R\cos(\omega t - \delta)$ is your answer.

   R – amplitude, ω – frequency, δ – phase shift

   $Re^{-\gamma t/2m}$ could be amplitude in damped case.

   $Re^{-\gamma t/2m}\cos(\mu t - \delta)$, μ – quasifrequency

   $T = \frac{2\pi}{\mu}$ – quasiperiod

   $T = \frac{2\pi}{\omega}$ – period

Ex. 1 A spring is stretched 4 in by a mass that weighs 8 lb. The mass is attached to a ~~dashp~~ dashpot that has a damping const of $\frac{1}{4}\,\frac{lb\text{-}sec}{ft}$, and is acted upon by an external force of $5\sin 2t$ lb. Find DE that governs this motion.

$$mu'' + \gamma u' + ku = F(t) \qquad u(0) = u_0, \quad u'(0) = u_0'$$

$$L = 4(in) = \frac{1}{3}(ft)$$

~~w = mg = 8 lb~~ ~~m = 8 lb~~

$$w = mg = 8\,(lb)$$

$$\gamma = \frac{1}{4}\,\frac{lb\cdot sec}{ft} \qquad m = \frac{8}{g}\,\frac{lb\cdot sec^2}{ft} = \frac{1}{4}(slugs)$$

$$mg = kL \Rightarrow k = \frac{mg}{L} = \frac{8\,(lb)}{\frac{1}{3}(ft)} = 24\left[\frac{lb}{ft}\right]$$

$$F(t) = 5\sin 2t\ (lb)$$

$$\Rightarrow \boxed{\frac{1}{4}u'' + \frac{1}{4}u' + 24u = 5\sin 2t}$$

Ex. 2 A mass of 1 kg is attached to a spring with $k = 3\,\frac{N}{m}$. A mass is acted upon by an external force of $2\sin t$ (N), and moves in a viscous fluid with resistance force $2u'$(N). Write DE, find gen. sol., find amplitude of the steady-state sol.

$\gamma u' = 2u'$

$$\left.\begin{array}{l} k = 3 \\ m = 1 \\ F = 2\sin t \\ \gamma = 2 \end{array}\right\} \Rightarrow u'' + 2u' + 3u = 2\sin t$$

$$u'' + 2u' + 3u = \underline{2\sin t}$$

$$r^2 + 2r + 3 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 12}}{2} = -1 \pm \sqrt{2}\, i$$

$$u_{hom.} = C_1 e^{-t} \cos\sqrt{2}t + C_2 e^{-t} \sin\sqrt{2}t$$

$$y_p = A\sin t + B\cos t$$

$$\Rightarrow u_{non.\,hom.}(t) = C_1 e^{-t}\cos\sqrt{2}t + C_2 e^{-t}\sin\sqrt{2}t + A\sin t + B\cos t$$

$$= C_1 e^{-t}\cos\sqrt{2}t + C_2 e^{-t}\sin\sqrt{2}t + R\cos(\omega t - \delta)$$

$$\boxed{R = \sqrt{A^2 + B^2}}$$

$$A\sin t + B\cos t = R\cos(\omega t - \delta)$$

$A = R\sin\delta$, $B = R\cos\delta$

$$\boxed{\omega = \sqrt{\frac{k}{m}}, \quad \tan\delta = \frac{B}{A}}$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$y_p' = A\cos t - B\sin t$$

$$y_p'' = -A\sin t - B\cos t$$

$$\Rightarrow -A\sin t - B\cos t + 2A\cos t - 2B\sin t + 3A\sin t + 3B\cos t = 2\sin t$$

$$\begin{cases} -A - 2B + 3A = 2 \\ -B + 2A + 3B = 0 \end{cases} \Rightarrow \begin{aligned} 2A - 2B &= 2 \\ 2A + 2B &= 0 \end{aligned}$$

$A - B = 1$

$A = -B$

$$\underline{A = \tfrac{1}{2} \quad B = -\tfrac{1}{2}}$$

$$u(t) = \underbrace{C_1 e^{-t}\cos\sqrt{2}t + C_2 e^{-t}\sin\sqrt{2}t}_{\text{transient}} + \underbrace{\frac{1}{\sqrt{2}}\cos(\omega t - \delta)}_{\text{steady-state}}$$

$\omega = \sqrt{3}$, $\delta = -\frac{\pi}{4}$

§6.1 Laplace Transform

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty} \int_a^A f(t)\,dt \quad \text{improper integral}$$

$$|f(t)| \le g(t) \text{ and } \int_a^\infty g(t) < \infty \Rightarrow \int_a^\infty f(t) < \infty.$$

Def. $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

is called a Laplace transform of f.

Thm. 1) f - piecewise continuous on $0 \le t \le A$

2) $|f(t)| \le Ke^{at}$, $t \ge M$

then $\mathcal{L}\{f(t)\} = F(s)$ exists for all $s > a$.

Properties:

1) $\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\}$

! 2) $\boxed{\mathcal{L}\{f\cdot g\} \ne \mathcal{L}\{f\}\cdot\mathcal{L}\{g\}}$

3) $\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = +\lim_{A\to\infty}\int_0^A e^{-st}\,dt =$

$= -\lim_{A\to\infty} \frac{e^{-st}}{s}\Big|_0^A = -\lim_{A\to\infty}\left(\frac{e^{-sA}-1}{s}\right) = \frac{1}{s}$

4) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $s > a$

5) $\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$

$\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0)$

$\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

__Example__ $y'' - y' - 6y = 2 \quad y(0) = 0,\ y'(0) = 0$

Strategy: Apply Laplace transform to both sides.

$$\mathcal{L}\{y'' - y' - 6y\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\underbrace{\mathcal{L}\{y\}}_{F(s)} = 0$$

$$\left[s^2F(s) - s\cancelto{0}{y(0)} - \cancelto{0}{y'(0)}\right] - \left[sF(s) - \cancelto{0}{y(0)}\right] - 6F(s) = 0$$

$$(s^2 - s - 6)F(s) = 2$$

$$F(s) = \frac{2}{s^2 - s - 6}$$