Math 214.002
Lecture 18

Example.

A mass weighing 2 lb stretches a spring 6 in. If the mass is pulled down an additional 3 in and then released, and if there is no damping, determine position of mass at any time t.

$u(t)$ – dispacement of mass from equilibrium

$mg = kL$ ← equilibrium condition

To find: $\boxed{mu'' + \gamma u' + ku = F(t)}$ $u(0) = u_0$, $u'(0) = u_0'$

$m = ?$ $m = \frac{w}{g} = \frac{2}{32} \frac{lb \cdot sec^2}{ft} = \boxed{\frac{1}{16}}$

$\gamma = ?$ $\gamma = 0$ (no damping)

$k = ?$ $k = \frac{mg}{L} = \frac{2\, lb}{\frac{1}{2} ft} = \boxed{4}$

$F(t) = ?$ $F(t) = 0$.

From problem: $w = 2\,lb$ $\quad$ $1\,ft = 12\,in$

$g = 32 \frac{ft}{sec^2}$

$L = 6in = \frac{1}{2} ft$

$u(0) = 3in = \frac{1}{4} (ft)$ ← initial displacement

$u'(0) = 0$ ← no initial velocity

$$\Rightarrow \left[\frac{1}{16}u'' + 4u = 0, \quad u(0) = \frac{1}{4}, \; u'(0) = 0\right]$$

$$u'' + 64u = 0$$

$$r^2 + 64 = 0 \qquad r = \pm 8i$$

$u(t) = A\cos 8t + B\sin 8t$

$u'(t) = -8A\sin 8t + 8B\cos 8t$

$u(0) = A = \frac{1}{4}$

$u'(0) = 8B = 0$ $\Rightarrow$ $\boxed{u(t) = \frac{1}{4}\cos 8t}$

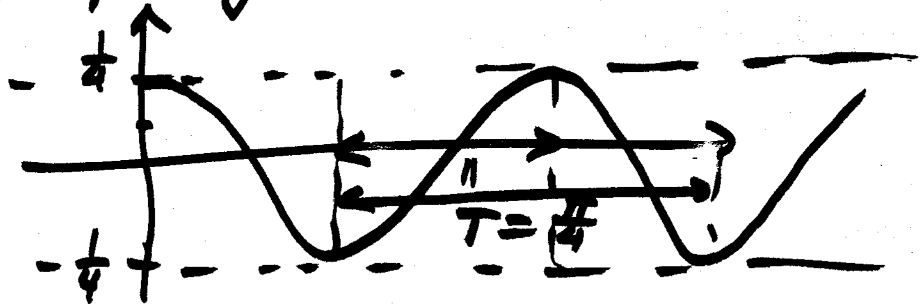
$$\boxed{u = R\cos(\omega_0 t - \delta)}$$

amplitude (R), natural frequency (ω_0), phase (δ)

Amplitude $= \frac{1}{4}$

period $= \frac{2\pi}{8} = \frac{\pi}{4}$

frequency $= 8$



Ex. A mass of 1 kg stretches a spring 1 m. The system has a damping const $= 6\,\frac{kg}{s}$. The mass starts at its equilibrium position with velocity $6\,\frac{m}{s}$. Take $g = 10\,\frac{m}{s^2}$

(a) Model this system

(b) What is the quasi-frequency?

$\underline{m}u'' + \underline{\gamma}u' + \underline{k}u = F(t)$, $u(0) = u_0$, $u'(0) = u_0'$

$m = 1\,kg$

$L = 1\,m \Rightarrow mg = kL \Rightarrow k = \frac{mg}{L} = 10\,\frac{kg \cdot \frac{m}{s^2}}{m} = 10\,\frac{kg}{s^2}$

$\gamma = 6\,\frac{kg}{s}$

$F = 0$, $u(0) = 0$

$u'(0) = 6\,\frac{m}{s}$

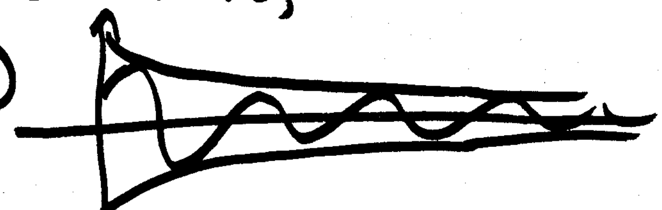
$$[u'' + 6u' + 10u = 0,\ u(0) = 0,\ u'(0) = 6]$$

$r^2 + 6r + 10 = 0$

$$r = \frac{-6 \pm \sqrt{36-40}}{2} = \frac{-6 \pm 2i}{2} = -3 \pm i$$

$u = C_1 e^{-3t}\cos t + C_2 e^{-3t}\sin t,$ $\lambda = -3$ $\mu = 1$

(underdamped case)



$u(0) = C_1 = 0$

$$u'(t) = -3C_1 e^{-3t}\cos t - C_1 e^{-3t}\sin t - 3C_2 e^{-3t}\sin t + C_2 e^{-3t}\cos t$$

$u'(0) = -3C_1 + C_2 = 6 \Rightarrow C_2 = 6$

$$\boxed{u = \underbrace{6e^{-3t}}_{\text{amplitude decreases with time}} \underbrace{\sin t}_{\cos(\frac{\pi}{2} - t)}}$$

$\mu = 1 \leftarrow$ quasi-frequency

$T_d = \frac{2\pi}{\mu} = 2\pi \leftarrow$ quasi-period

§3.8 Forced vibrations.

$\boxed{F(t) = F_0 \cos \omega t}$

$$mu'' + \gamma u' + ku = F(t), \quad F(t) \neq 0.$$

Case 1. Damped forced vibrations.

$$u = \underbrace{C_1 u_1 + C_2 u_2}_{} + y_p \qquad \gamma \neq 0 \Rightarrow$$

$$u_{hom} = C_1 e^{-\frac{\gamma t}{2m}} \cos \omega_0 t + C_2 e^{-\frac{\gamma t}{2m}} \sin \omega_0 t$$

$$y_p = A\cos\omega t + B\sin\omega t = R\cos(\omega t - \delta)$$

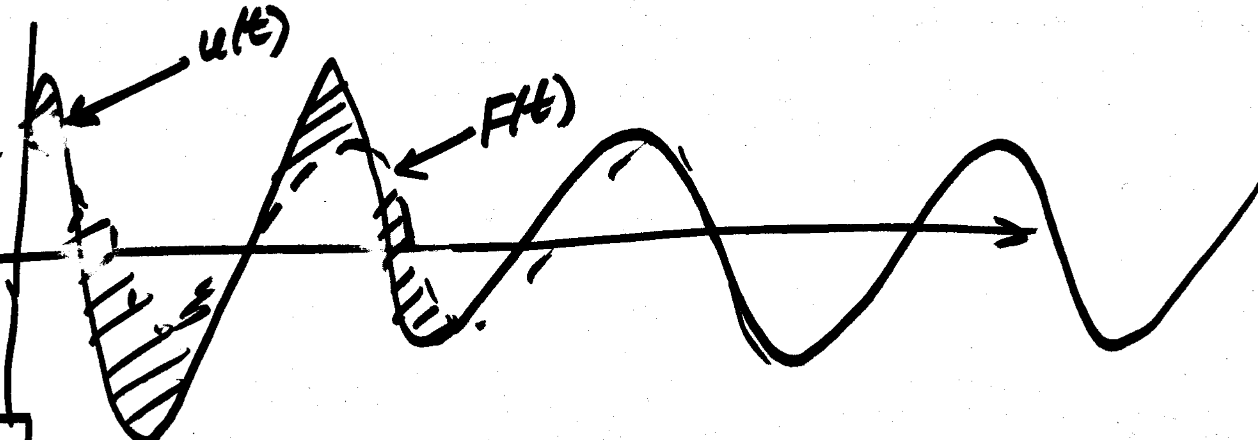
$$u = \underbrace{C_1 e^{-\frac{\gamma t}{2m}} \cos \omega_0 t + C_2 e^{-\gamma t/2m} \sin \omega_0 t}_{\text{transient solution}} + \underbrace{R\cos(\omega t - \delta)}_{\text{Steady-state solution}}$$

transient solution $u_c(t) \to 0$ as $t \to \infty$

Steady-state Solution

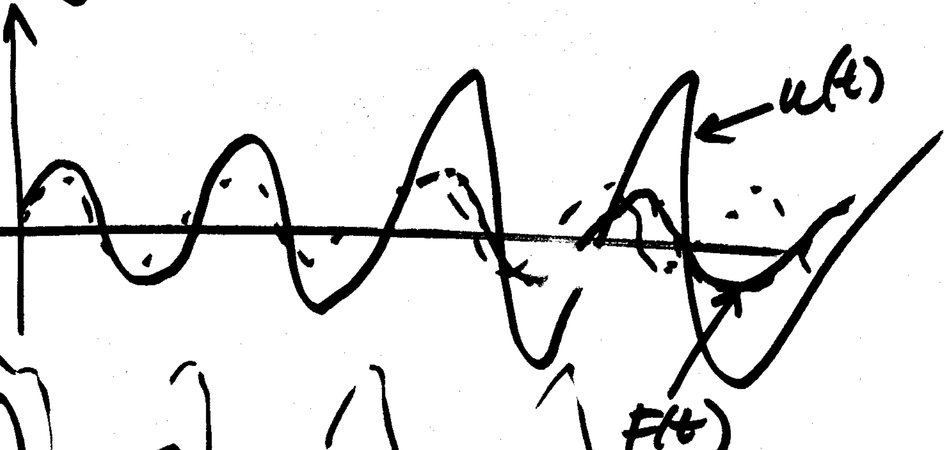
forced response

$\boxed{\frac{\omega}{\omega_0} < 1}$: u amplitude will stay almost const

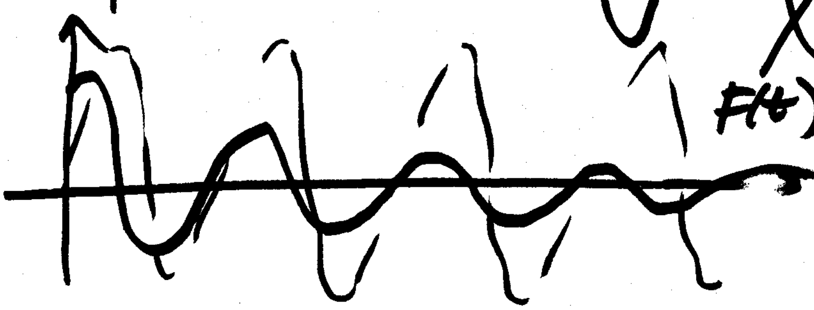


$\boxed{\omega = \omega_0}$

amplitude will grow



$\frac{\omega}{\omega_0} > 1$



Case 2. Undamped forced vibrations.

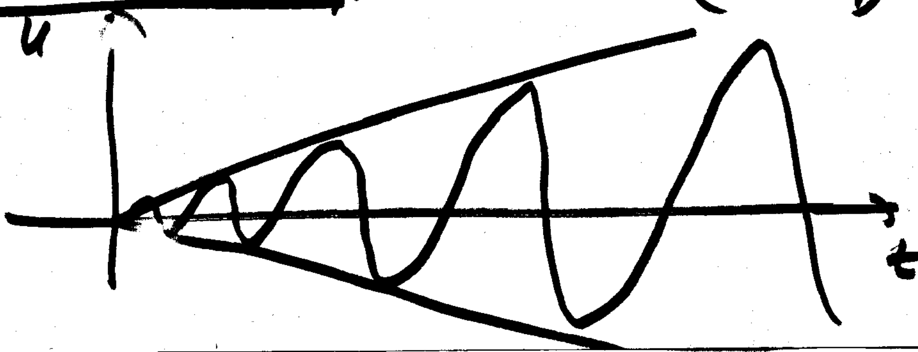
$$mu'' + ku = F_0 \cos \omega t$$

$$u = \underline{C_1 \cos \omega_0 t + C_2 \sin \omega_0 t} + (A \cos \omega t + B \sin \omega t) \cdot t^s$$

if $\omega = \omega_0 \Rightarrow s = 1$

if $\omega \neq \omega_0 \Rightarrow s = 0$

$\boxed{\omega = \omega_0 \quad \text{resonance}}$: $u = R_1 \cos(\omega_0 t - \delta_1) + \underbrace{t R_2 \cos(\omega t - \delta_2)}_{\text{grows to } \infty \text{ as } t \to \infty}$



$\underline{\omega \neq \omega_0}$

$$u = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

$$\left.\begin{array}{l} u(0)=0 \\ u'(0)=0 \end{array}\right\} \Rightarrow u = \left[\frac{2F_0}{m(\omega_0^2-\omega^2)} \sin \frac{(\omega_0-\omega)t}{2}\right] \sin \frac{(\omega_0+\omega)t}{2}$$

Slowly oscillation

rapidely oscillating