Math 214.002.

Lecture 17.

We know: $\underline{ay'' + by' + cy = g(t)}$

$g(t) \neq 0 \rightarrow$ undetermined coeffs
<u>OR</u> Variation of parameters

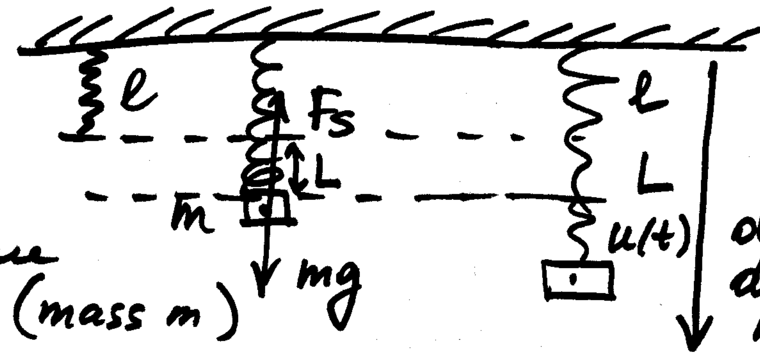
Modeling with 2nd order equations.

§3.8. Mechanical & Electrical Vibrations.

<u>Spring-mass</u>:

ℓ - natural length of spring

L - elongation due to gravity (mass m)



displacement due to motion

$\underline{F = ma = 0}$ in equilibrium

$\vec{F_s}$ & $\vec{mg}$
Spring force, gravity

$\boxed{mg = F_s}$ $\quad F_s = kL$

$\boxed{mg = kL}$ $\quad$ Hooke's law

Balance of forces for motion of the spring-mass system:

$u(t)$ - displacement

$u'(t)$ - velociy, $u''(t)$ - acceleration

$F = ma$ $\qquad F = mg + F_s + F_d + F$

$F = mu''$

Spring force, damping force, external force

$$mu'' = F = mg - k(\underbrace{L+u}_{\text{total dispacement}}) \mp \gamma u' + F$$

γ - damping constant

$$mu'' = \underbrace{mg - kL} - ku - \gamma u' + F$$

$$mu'' = -ku - \gamma u' + F$$

$$\boxed{mu'' + \gamma u' + ku = F} \quad u(0) = u_0, \; u'(0) = u_0'$$

initial position & velocity

1. Free undamped vibration: $mu'' + ku = 0$

$\downarrow F = 0$ $\quad \gamma = 0$

$$mr^2 + k = 0$$

$$r^2 = -\frac{k}{m} \quad r = \pm i \underbrace{\sqrt{\frac{k}{m}}}_{\omega_0}$$

natural frequency

$$u = A\cos\omega_0 t + B\sin\omega_0 t$$

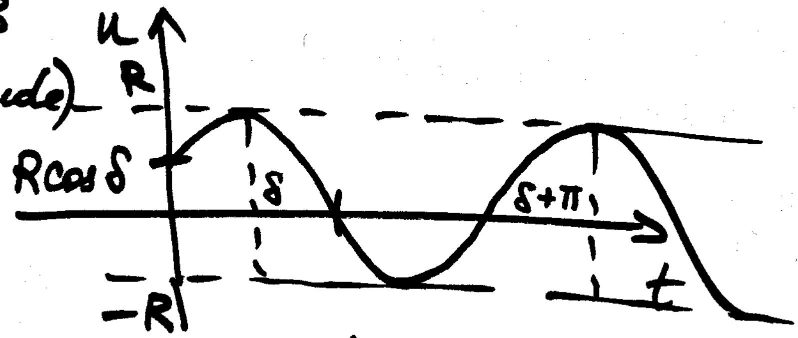
$$u = \boxed{R\cos(\omega_0 t - \delta)}$$

$$= \underbrace{R\cos\delta}_{A}\cos\omega_0 t + \underbrace{R\sin\delta}_{B}\sin\omega_0 t$$

$R = \sqrt{A^2 + B^2}$ (amplitude)

$\tan\delta = \frac{B}{A}$

(δ - phase)



$T = \frac{2\pi}{\omega_0} = 2\pi\left(\frac{k}{m}\right)^{-1/2}$ - natural period

$$\omega_0 = \sqrt{\frac{k}{m}}$$

2. Damped vibrations (free) $\to F = 0$.

$$mu'' + \gamma u' + ku = 0$$

$$mr^2 + \gamma r + k = 0$$

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

1) $\gamma^2 - 4km > 0$ $\quad u = Ae^{r_1 t} + Be^{r_2 t}, \; r_1, r_2 < 0$

$u \to 0$ as $t \to \infty$

2) $\gamma^2 - 4km = 0$ $\quad u = Ae^{rt} + Bte^{rt}, \; r = -\frac{\gamma}{2m} < 0$

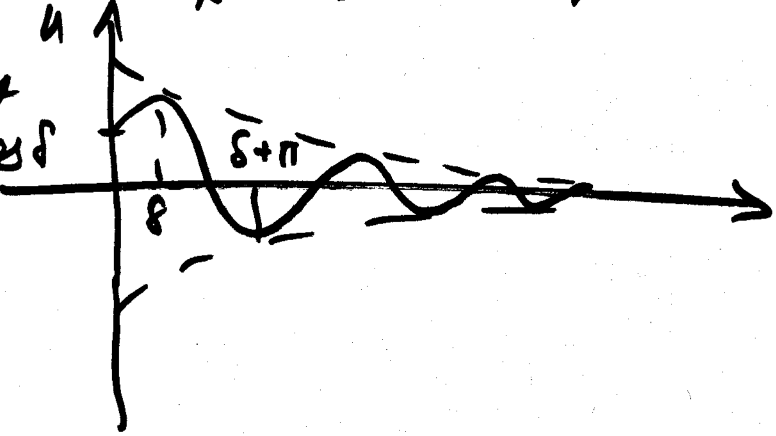
3) $\gamma^2 - 4km < 0$

$$u = e^{-\gamma t/2m}(A\cos\mu t + B\sin\mu t) \xrightarrow[t\to\infty]{} 0$$

$$\mu = \frac{\sqrt{4km - \gamma^2}}{2m}$$

$$u = Re^{-\gamma t/2m}\cos(\mu t - \delta) \text{ - not periodic}$$

μ - <u>quasifrequency</u>



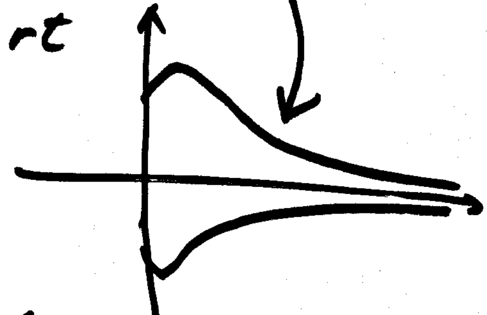
$\gamma^2 - 4km < 0$

$\boxed{\gamma^2 \sim 4km}$

$\gamma = 2\sqrt{km}$ - critical damping

switch from oscillations to

$$u = Ae^{rt} + Bte^{rt}$$



<u>Ex</u>. A mass of 3 kg attached to a spring with $k = 12\frac{N}{m}$.
What value of damping coefficient will make the system critically damped?

$$\gamma = 2\sqrt{km} = 2\cdot\sqrt{12\left[\tfrac{N}{m}\right]\cdot 3[kg]} = 2\sqrt{4\cdot 3^2} = \underline{12}$$

$$2\cdot\sqrt{6^2}$$