Math 214.002

Lecture 16

§3.6. Method of undetermined coefficients.

$$y'' + p(t)y' + q(t)y = g(t) \neq 0$$

Special type: $\boxed{ay'' + by' + cy = g(t)}$ non-hom. with const. coefficients

$$ay'' + by' + cy = 0 \leftarrow \text{corresp. hom. eqn.}$$

<u>Thm</u>: $\begin{bmatrix} \text{gen. sol.} \\ \text{non-hom. eqn.} \end{bmatrix} = \begin{bmatrix} \text{gen. sol.} \\ \text{hom. eqn.} \end{bmatrix} + \begin{bmatrix} \text{particular} \\ \text{sol.} \\ \text{of non-hom. eqn.} \end{bmatrix}$

$$y_{non.hom.}(t) = y_{hom.}(t) + y_p$$

<u>To get y_p:</u>

1) <u>Method of undeterm. coeffs.</u>
2) Method of variation of parameters.

Starting with $ay'' + by' + cy = g(t)$:

1) Solve $ay'' + by' + cy = 0$
(Find roots r_1, r_2 of char. equation $\boxed{ar^2 + br + c = 0}$ (*))

2) Determine the type of function $\underline{g(t)}$; find $\underline{y_p(t)}$.

$g(t)$	$y_p(t)$
① "exp" type: $e^{\alpha t}$ $\longrightarrow$	$a \cdot e^{\alpha t} \cdot t^s$, $s =$ multiplicity of α as a root of (*)
② "trig" type: $\sin\beta t$ or $\cos\beta t$ $\longrightarrow$	$t^s(a\sin\beta t + b\cos\beta t)$, $s =$ mult. of $\alpha + i\beta$ in (*)

③ "poly" type: $a_n t^n + a_{n-1} t^{n-1} + .. + a_0 \longrightarrow (b_n t^n + \ldots + b_0) t^s$,

s = mult. of ($r=0$) in eqn. ⊛

To generalize:

$P_n(t) e^{\alpha t} \longrightarrow Q_n(t) e^{\alpha t} \cdot t^s$

P_n: poly of deg $= n$; Q_n: poly of deg $= n$

$P_n(t) \cos\beta t \longrightarrow Q_n(t)(a \sin\beta t + b\cos\beta t) \cdot t^s$

Q_n: poly of deg $= n$

$P_n(t) e^{\alpha t} \sin\beta t \longrightarrow Q_n(t) \cdot e^{\alpha t}(a \sin\beta t + b \cos\beta t) \times t^s$

Q_n: poly of deg $= n$

Sum of several terms:

$g(t) = g_1(t) + \ldots + g_n(t)$

$e^{\alpha t} + \cos\beta t + P_n(t) \longrightarrow t^{s_1} a e^{\alpha t} + (b\cos\beta t + c \cdot \sin\beta t) \times t^{s_2} + Q_n(t) \cdot t^{s_3}$

Examples.

1. Type 1 ("exp type"): $y'' - y' - 2y = \boxed{3e^{-t}}$

$y'' - y' - 2y = 0$

$r^2 - r - 2 = 0 \quad (r-2)(r+1) = 0$

$r = 2 \quad r = -1$

$y_{hom.}(t) = C_1 e^{2t} + C_2 e^{-t}$

$y_p(t) = \underline{\underline{a \cdot e^{-t} \cdot (t)}}$ to avoid duplication with term $\underline{\underline{C_2 e^{-t}}}$ in hom. equation

$$\left.\begin{aligned} y_p &= ate^{-t} \\ y_p' &= ae^{-t} - ate^{-t} \\ y_p'' &= -ae^{-t} - ae^{-t} + ate^{-t} \end{aligned}\right\} \Rightarrow$$ plug into the equation

$$-2ae^{-t} + \cancel{ate^{-t}} - ae^{-t} + \cancel{ate^{-t}} - \cancel{2ate^{-t}} = 3e^{-t}$$

$$-3ae^{-t} = 3e^{-t} \qquad \boxed{a=-1}$$

$$y_p = -te^{-t} \Rightarrow \boxed{y_{non.hom.}(t) = C_1e^{2t} + C_2e^{-t} - te^{-t}}$$

2. $y'' - y' - 2y = 2t + t^2e^t$

Find y_p: $y_p = (a_0t + b_0) + (a_1t^2 + b_1t + c_1)e^t$

$r=2,\ r=-1$

$y_{hom.} = C_1e^{2t} + C_2e^{-t}$

3. $y'' + 4y' + 3y = 3e^{-t} + 5$

Find y_p.

4. $y'' + 4y' + 4y = 2e^{-2t}$ $\qquad r=-2$

$y_p = at^2e^{-2t}$ $\qquad y = C_1e^{-2t} + C_2te^{-2t}$

5. $y'' + 4y' + 4y = 2te^{-2t}$

$y_p = (at+b)e^{-2t} \cdot t^2$

6. $y'' + 2y' - 3y = 5\sin 3t$

$r^2 + 2r - 3 = 0$

$(r+3)(r-1) = 0 \qquad r=-3\ r=1 \qquad y_{hom} = C_1e^t + C_2e^{-3t}$

$y_p = a\sin 3t + b\cos 3t$

$y_p' = 3a\cos 3t - 3b\sin 3t$

$y_p'' = -9a\sin 3t - 9b\cos 3t$

$$\hookrightarrow \ -9a\sin 3t - 9b\cos 3t + 6a\cos 3t - 6b\sin 3t$$
$$-3a\sin 3t - 3b\cos 3t = 5\sin 3t$$

$$(-9a - 6b - 3a)\sin 3t = 5\sin 3t$$
$$(-9b + 6a - 3b)\cos 3t = 0$$

Want this to hold for all t values $\Rightarrow$

$$\begin{cases} -9a - 6b - 3a = 5 \\ -9b + 6a - 3b = 0 \end{cases} \qquad \begin{cases} -12a - 6b = 5 \\ 6a - 12b = 0 \end{cases}$$

$$a = 2b$$
$$-24b - 6b = 5$$
$$-30b = 5 \quad b = -\tfrac{1}{6}$$
$$a = -\tfrac{1}{3}$$

$$y_p = -\tfrac{1}{3}\sin 3t - \tfrac{1}{6}\cos 3t \Leftarrow$$

$$\left[y_{non.hom.} = C_1 e^{t} + C_2 e^{-3t} - \tfrac{1}{3}\sin 3t - \tfrac{1}{6}\cos 3t\right]$$

7. $y'' + 9y = 5\cos 3t + e^{t}\sin 3t$

$\underline{y_p = ?}$

8. $y'' + 9y = \underline{5t^2}\,\underline{\cos 3t}$

$$y_p = (at^2 + bt + \underline{\underline{c}})(d\cos 3t + e\sin 3t)\,t$$

9. $y'' + 4y' = \underline{2} + \overset{P_1}{3t}\,\underline{\underline{e^{-4t}}} + \underline{\underline{\underline{2\cos 4t}}}$

$r = 0 \quad r = -4$

$$y_{hom.} = C_1 + C_2 e^{-4t}$$

$$y_p = \underline{a}\,t + \underbrace{(bt + c)}_{Q_1}e^{-4t}\cdot t + (d\cos 4t + e\sin 4t)$$