Math 214.002
Lecture 15.

$ay'' + by' + cy = 0.$ $\quad ar^2 + br + c = 0$

char. polynomial

3 cases:

1) r_1, r_2 - roots of char. poly

$r_1 \neq r_2$ real $\quad \boxed{y = C_1 e^{r_1 t} + C_2 e^{r_2 t}}$

2) r_1, r_2 - complex $\quad r_1 = \lambda + i\mu$

$r_2 = \lambda - i\mu$

$$\boxed{y = C_1 e^{\lambda t} \cos\mu t + C_2 e^{\lambda t} \sin\mu t}$$

3) $r = r_1 = r_2$ real, $\quad r = -\frac{b}{2a}$

$$\boxed{y = C_1 e^{rt} + C_2 t e^{rt}}$$

Examples. $\quad y'' + 6y' + 9y = 0 \quad y(0) = 1,\ y'(0) = 2$

$r^2 + 6r + 9 = 0$

$(r+3)^2 = 0 \quad r = -3$ repeated root

$y = C_1 e^{-3t} + C_2 t e^{-3t}, \quad y' = -3C_1 e^{-3t} + C_2 e^{-3t} - 3C_2 t e^{-3t}$

$y(0) = C_1 = 1$

$y'(0) = -3C_1 + C_2 = 2 \Rightarrow -3 + C_2 = 2$

$C_2 = 5$

$$\boxed{y = e^{-3t} + 5te^{-3t}}$$

Reduction of order

a method for finding y_2 if y_1 is known.
Works for any equation $y''+p(t)y'+q(t)y=0$ (*)
Starting with $y''+py'+qy=0$ and
given $y_1 \neq 0$ ⇐ nonzero solution to (*),
let us denote $\boxed{y_2(t)=v(t)y_1(t)}$.

Plug into (*):

$$y_2' = v'y_1 + vy_1'$$
$$y_2'' = v''y_1 + \underline{v'y_1'} + \underline{v'y_1'} + vy_1''$$
$$= v''y_1 + 2v'y_1' + vy_1''$$

$$(\underline{v''y_1} + \underline{\underline{2v'y_1'}} + vy_1'') + (\underline{\underline{pv'y_1}} + pvy_1') + qvy_1 = 0$$

$$v''y_1 + (2y_1' + py_1)v' + \underbrace{(y_1''+py_1'+qy_1)}_{0}v = 0$$

because y_1 – solution to (*).

$$y_1 v'' + (2y_1' + py_1)v' = 0$$

$$\left.\begin{aligned} u &= v' \\ u' &= v'' \end{aligned}\right\} \Rightarrow \left[y_1 u' + (2y_1' + py_1)u = 0 \right]$$

reduced 2nd order eqn (*)
to a 1st order eqn.

⇒ Solve it for u and get $\underline{v = \int u\,dt}$

Ex. $t^2y'' - 4ty' + 6y = 0$, $t>0$ $y_1 = t^2$
Find $y_2(t)$ using reduction of order.

Known: If $y_2 = v(t)y_1(t)$, then
$v(t)$ satisfies $y_1v'' + (2y_1' + py_1)v' = 0$.

Make substitution $u = v'$

$$\boxed{y_1 u' + (2y_1' + py_1)u = 0}$$

In our example, $y_1 = t^2$, $y_1' = 2t$

$$t^2 y'' - 4ty' + 6y = 0 \Rightarrow y'' - \frac{4}{t}y' + \frac{6}{t^2}y = 0$$

Standard form

$$p = -\frac{4}{t}$$

$$\Rightarrow \quad t^2 u' + \underbrace{\cancel{\left(4t - \frac{4}{t}\cdot t^2\right)}}_{0} u = 0$$

$$\Rightarrow t^2 u' = 0 \quad \text{if } t \neq 0 \quad u' = 0.$$

$$u = C$$

$$u = v' = C$$

$$v = Ct + C_1 \quad \Rightarrow \quad y_2 = v \cdot y_1 =$$

$$= (Ct + C_1)t^2$$

$$= Ct^3 + C_1 t^2$$

We had: $y_1 = t^2$

We computed: $y_2 = Ct^3 + C_1 t^2$

easiest choice: $C = 1$, $C_1 = 0$ $\boxed{y_2 = t^3}$

General solution:

$$y = C_0 y_1 + C_2 y_2$$

$$\boxed{y = C_0 t^2 + C_2 t^3}$$

Check: y_1, y_2 – lin. independent

$$W(y_1, y_2) = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} = 3t^4 - 2t^4 = t^4$$

Alternative method for finding y_2

(Abel formula):

$$ay'' + by' + cy = 0 \qquad r_1 = r_2 = -\frac{b}{2a} \text{ repeated root}$$

$$y_1 = e^{-\frac{bt}{2a}}$$

Find y_2:

$$y'' + \frac{b}{a} y' + \frac{c}{a} y = 0$$

By Abel formula: $W(y_1, y_2) = e^{-\frac{bt}{a}}$

$\left(W = Ce^{-\int p(t)\,dt} \right)$

By definition: $W = y_1 y_2' - y_2 y_1'$ $\qquad \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$\Rightarrow y_1 \cdot y_2' - y_1' \cdot y_2 = Ce^{-\frac{bt}{a}}$$

$$y_1 = e^{-bt/2a}, \quad y_1' = -\frac{b}{2a} e^{-bt/2a}$$

$$\Rightarrow e^{-bt/2a} y_2' + \frac{b}{2a} e^{-bt/2a} \cdot y_2 = Ce^{-\frac{bt}{a}}$$

$$y_2' + \frac{b}{2a} y_2 = Ce^{-\frac{bt}{2a}} \qquad \mu = e^{+\frac{bt}{2a}}$$

$$\left(e^{\frac{bt}{2a}} y_2\right)' = C$$

$$e^{bt/2a} y_2 = Ct + C_1 \Rightarrow \underline{y_2 = Cte^{-\frac{bt}{2a}} + C_1 e^{-\frac{bt}{2a}}}$$

Example. $t^2 y'' - 4ty' + 6y = 0$, $\underline{y_1 = t^2}$

Standard form: $y'' \underbrace{- \frac{4}{t}}_{p(t)} y' + \frac{6}{t^2} y = 0$

Abel formula:

$$W = Ce^{+\int \frac{4}{t} dt} = Ct^4$$

Using definition: $W = y_1 \cdot y_2' - y_1' \cdot y_2 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Putting together: $y_1 \cdot y_2' - y_1' \cdot y_2 = Ct^4$

$$t^2 y_2' - 2t y_2 = Ct^4$$

$p = -\frac{2}{t}$

$$y_2' - \frac{2}{t} y_2 = Ct^2$$

$$\mu = e^{\int p(t)dt} = e^{-\int \frac{2}{t}dt} = e^{-2\ln t} = t^{-2}$$

$$(t^{-2} y_2)' = C$$

$$\frac{y_2}{t^2} = Ct + C_1$$

$$\underline{y_2(t) = Ct^3 + C_1 t^2}$$

$C_1 = 0$ $C = 1$ for instance

$$y_2 = t^3$$

§3.6 Nonhomogeneous equations.

$$y'' + p(t)y' + q(t)y = g(t) \quad (g(t) \neq 0)$$

(Will Start with const. coeff. case:

$$ay'' + by' + cy = g(t))$$

Fact: y_1, y_2 - solutions to $y'' + p(t)y' + q(t)y = g(t)$ (*)
then $y_1 - y_2$ solves $y'' + p(t)y' + q(t)y = 0$. (**)

Fact 2 (follows from Fact 1):

if $y = C_1 y_1(t) + C_2 y_2(t)$ - gen. solution to homogeneous equation (**), then

$y_{non.hom.}(t) = C_1 y_1(t) + C_2 y_2(t) + Y_p(t)$ - gen. solution to (*) for any special solution $Y_p(t)$ of (*)

Reason: $y_1 - y_2$ solves (**)

$\Rightarrow y_1 - y_2 = c_1 y_1(t) + c_2 y_2(t)$

Choose $y_2 = y_p(t)$ some particular solution

Then for any $y_1(t)$, we get

$y_1(t) = y_p(t) + c_1 y_1(t) + c_2 y_2(t)$

$$\left\{\begin{array}{c}\text{gen. sol.}\\ \text{of non. hom. eqn.}\end{array}\right\} = \left\{\begin{array}{c}\text{gen. sol.}\\ \text{of hom. eqn.}\end{array}\right\} + \left\{\begin{array}{c}\text{particular}\\ \text{sol. of}\\ \text{non. hom.}\\ \text{eqn.}\end{array}\right\}$$

The whole task is to find $\underline{\underline{y_p}}$.

Ex. 1. $y'' - 2y' - 3y = \underline{\underline{3e^{2t}}}$

$y'' - 2y' - 3y = 0$

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0 \quad \begin{array}{l} r = 3 \\ r = -1 \end{array} \Rightarrow y_{hom.} = C_1 e^{3t} + C_2 e^{-t}$

$$\left.\begin{array}{l} y_p = Ae^{2t} \\ y_p' = 2Ae^{2t} \\ y_p'' = 4Ae^{2t} \end{array}\right\} \Rightarrow \begin{array}{l} \text{plug in } y'' - 2y' - 3y = 3e^{2t} \\ (\cancel{4A} - \cancel{4A} - 3A)\cancel{e^{2t}} = 3\cancel{e^{2t}} \\ -3A = 3 \quad A = -1 \end{array}$$

We found $y_p = -e^{2t}$ that solves our non. hom. equation.

$y_{non.hom.}(t) = y_{hom.} + y_p$

$= \underline{C_1 e^{3t} + C_2 e^{-t} - e^{2t}}$

Word of caution: if $r = 2$ was a root of char. equation associated with hom. eqn. in this problem, $y_p = Ae^{2t}$ wouldn't work.

Have to modify $y_p = A\underline{\underline{t}}e^{2t}$