Math 214 002

Lecture 14.

$ay'' + by' + cy = 0$

$ar^2 + br + c = 0$ characteristic equation

1) r_1, r_2 - real roots, $r_1 \neq r_2$

   $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ - gen. solution

$$\left.\begin{array}{l} 2)\ r_1 = r_2 \\ 3)\ r_1 = \lambda + i\mu \\ \quad r_2 = \lambda - i\mu \end{array}\right\} \text{today}$$

§3.4 Complex roots.

$ar^2 + br + c = 0 \qquad r_1 = \lambda + i\mu$

$\qquad\qquad\qquad\qquad\quad r_2 = \lambda - i\mu$

$y_1 = e^{r_1 t} = e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t} = \underline{e^{\lambda t}(\cos\mu t + i\sin\mu t)}$

$y_2 = e^{r_2 t} = e^{(\lambda - i\mu)t} = e^{\lambda t} e^{-i\mu t} = \underline{e^{\lambda t}(\cos\mu t - i\sin\mu t)}$

$[e^{it} = \cos t + i\sin t]$ Euler formula

$e^{i\mu t} = \cos\mu t + i\sin\mu t$

$y_1 + y_2 = 2e^{\lambda t}\cos\mu t = 2u(t)$

$y_1 - y_2 = (2i)e^{\lambda t}\sin\mu t = (2i)v(t)$

$$\left.\begin{array}{l} u(t) = e^{\lambda t}\cos\mu t \\ v(t) = e^{\lambda t}\sin\mu t \end{array}\right\} \text{also solutions to } ay'' + by' + cy = 0$$

$$W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t & \lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t \end{vmatrix}$$

$$= \cancel{\lambda e^{2\lambda t}\cos\cdot\sin} + \mu e^{2\lambda t}\cos^2 \cancel{- \lambda e^{2\lambda t}\cos\sin} + \mu e^{2\lambda t}\sin^2$$

$$= \mu e^{2\lambda t}\underbrace{(\cos^2+\sin^2)}_{=1} = \mu e^{2\lambda t} \neq 0$$

$\{u,v\}$ - independent

General solution: $y(t) = C_1 u(t) + C_2 v(t)$

$\{u,v\}$ - fund. set of solutions

<u>Conclusion</u>: $ay'' + by' + cy = 0$

$r_1, r_2 = \lambda \pm i\mu$ - complex roots of char. eqn

$$\Rightarrow \boxed{y(t) = C_1 e^{\lambda t}\cos\mu t + C_2 e^{\lambda t}\sin\mu t}$$

is the general solution

<u>Ex.</u> $y'' - 2y' + 2y = 0$

$r^2 - 2r + 2 = 0 \qquad r = \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$\lambda = 1 \quad \mu = 1$

$$y = C_1 e^t \cos t + C_2 e^t \sin t$$

Ex. $y'' + 4y = 0 \quad \underline{y(0)=0,\ y'(0)=1}$

$r^2 + 4 = 0$ $\qquad \left(\begin{array}{l} y'' + 4y' = 0 \\ r^2 + 4r = 0 \end{array} \right)$

$r = \pm 2i$

$\left.\begin{array}{l} \lambda = 0 \\ \mu = 2 \end{array}\right\} \Rightarrow y = C_1 \cos 2t + C_2 \sin 2t$

$y' = -2C_1 \sin 2t + 2C_2 \cos 2t$

$y(0) = \underline{C_1 = 0}$

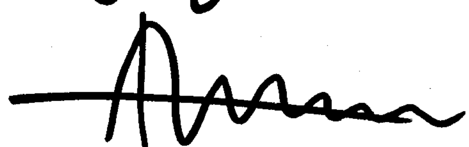
$y'(0) = 2C_2 = 1 \Rightarrow C_2 = \frac{1}{2} \Rightarrow \boxed{y = \frac{1}{2}\sin 2t}$

Ex. $y'' + 2y' + 2y = 0 \quad y(0) = 2,\ y'(0) = 3$

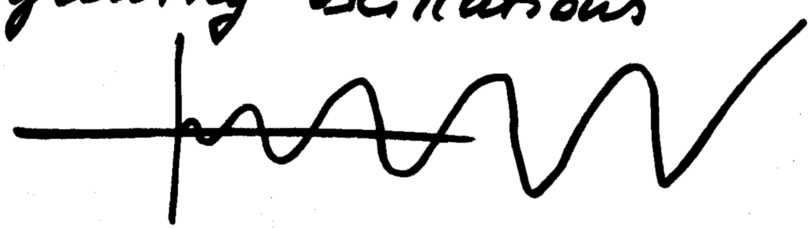
<u>Geometric Behavior</u>:

$y = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t$

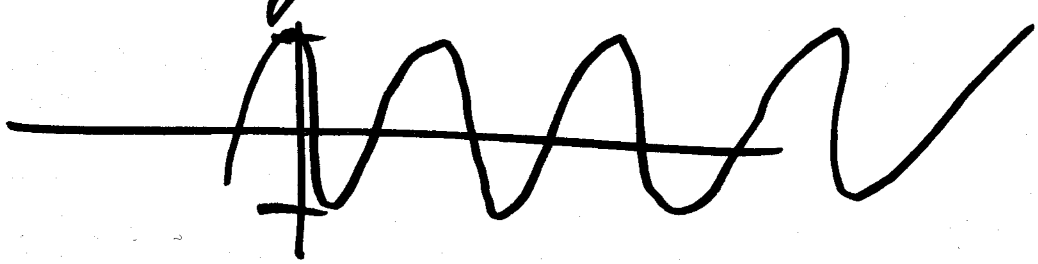
if $\lambda < 0 \Rightarrow$ decaying oscillations



If $\lambda > 0 \Rightarrow$ growing oscillations



$\lambda = 0 \Rightarrow$ steady oscillations



Last case: $r_1 = r_2 = r$ (Chapter 3.5)

$ar^2 + br + c = 0$ ← repeated roots

$$r = -\frac{b}{2a}$$

$y_1 = e^{rt} = e^{-bt/2a}$ ← one solution

How do we find the second one?

Choose $y_2 = \underbrace{\underline{v(t)}}_{\text{unknown}}\, \underbrace{y_1(t)}_{\text{known}} = v \cdot e^{-bt/2a}$

When (for which v) is y_2 a solution to

⊛ $ay'' + by' + cy = 0$? (given that $e^{-bt/2a}$ is a solution)

Plug y_2 into ⊛:

$$y_2 = v e^{-bt/2a}$$

$$y_2' = v' e^{-bt/2a} - \frac{b}{2a} v e^{-bt/2a}$$

$$y_2'' = v'' e^{-bt/2a} - \frac{2b}{2a} v' e^{-bt/2a} + \left(\frac{b}{2a}\right)^2 v e^{-bt/2a}$$

$\underline{ay_2''} + by_2' + cy_2 = 0$

$$e^{-bt/2a}\left(av'' \cancel{- bv'} + \frac{b^2}{4a} v\right) + e^{-bt/2a}\left(\cancel{bv'} - \frac{b^2}{2a} v\right) + e^{-bt/2a} \cdot (cv) = 0$$

$$\cancel{e^{-bt/2a}}\left(av'' + \frac{b^2 - 2b^2 + 4ac}{4a} v\right) = 0$$

$$av'' - \frac{\cancel{b^2 - 4ac}^{\,0}}{4a} v = 0$$

$\boxed{\text{if } r_1 = r_2: \quad b^2 - 4ac = 0}$

$$v''=0 \quad v'=c \quad v=ct+c_1$$

$$y_2 = v\cdot y_1 = (ct+c_1)y_1 = c\boxed{ty_1}+\underline{c_1y_1}$$

Conclusion: if $\begin{cases} y_1 = e^{-bt/2a} & \text{- first sol.} \\ y_2 = te^{-bt/2a} & \text{- second sol.} \end{cases}$ $\leftarrow$ fund. set of solutions

$$\boxed{y = C_1 te^{-bt/2a} + C_2 e^{-bt/2a}}$$

---

① $y''+2y'=0 \quad y(0)=0,\ y'(0)=2$

$r^2+2r=0$

$r(r+2)=0 \quad r=0,\ r=-2$

$y = C_1e^{0t}+C_2e^{-2t} = C_1+C_2e^{-2t}$

$y' = -2C_2e^{-2t}$

$y(0) = C_1+C_2=0$

$y'(0) = -2C_2 = 2$ $\Longrightarrow$ $C_1=1$, $C_2=-1$

$\underline{y = 1-e^{-2t}}$

② $\begin{cases} (t+4)y'' + \dfrac{1}{\sin t}\cdot y' + \ln|t-1|\,y = e^t \\ y(\underline{-2}) = 4 \end{cases}$ where is solution certain to exist?

$$y'' + \boxed{p(t)}y' + \boxed{q(t)}y = \boxed{g(t)}$$

$\pi \sim 3.14$