Math 214.002

Lecture 13

We know: $ay''+by'+cy=0$

$ar^2+br+c=0$

if r_1, r_2 - roots of char. eqn. $r_1 \neq r_2$ real

$\Rightarrow \underline{y = C_1 e^{r_1 t} + C_2 e^{r_2 t}}$

By Thm 4 (last time) if $W(y_1, y_2)(t_0) \neq 0$

$\Rightarrow y = C_1 y_1(t) + C_2 y_2(t)$ includes all solutions to equation $y''+p(t)y'+q(t)y=0$.

($\{y_1, y_2\}$ - fund. set of solutions).

<u>Question:</u> if $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$ $r_1 \neq r_2$ (real).

Prove that $\{y_1, y_2\}$ - fund. set of solutions.

<u>To show:</u> $\underline{W(y_1, y_2)(t_0) \neq 0 \text{ for some } t_0}$.

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}(t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} =$$

$$= r_2 e^{r_1 t} e^{r_2 t} - r_1 e^{r_1 t} e^{r_2 t} =$$

$$= \underbrace{(r_2 - r_1)}_{\neq 0 \text{ since } r_2 \neq r_1} e^{(r_1+r_2)t} \neq 0 \text{ for all } t$$

<u>Ex. 1</u> What is $W(y_1, y_2)$ if $y_1 = e^{2x}$, $y_2 = 2e^{2x}$

$$\begin{vmatrix} e^{2x} & 2e^{2x} \\ 2e^{2x} & 4e^{2x} \end{vmatrix} = (4-4)e^{4x} \equiv 0$$

$\Rightarrow \{y_1, y_2\}$ cannot form a fundam. set.

Ex. 2. If $W(f,g) = 3e^{4t}$ and $f(t) = e^{2t}$
Find $g(t)$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

§3.3. Linear Independence and Wronskian.

Def. $f(t), g(t)$ linearly dependent in I if there are C_1, C_2 not both zero s.t.
$C_1 f(t) + C_2 g(t) = 0$ for all t in I.

if $C_1 f(t) + C_2 g(t) = 0$

$$f(t) = -\frac{C_2}{C_1} g(t)$$

Thm 1. If $W(f,g)(t_0) \neq 0$ (t_0 belongs to interval I) then f, g - linearly independent on I.
Moreover, if f, g - lin. dependent $\Rightarrow$ $W(f,g) = 0$ for all t in I.

Example: $\{e^{r_1 t}, e^{r_2 t}\}$ $r_1 \neq r_2 \Rightarrow$ lin. independent in $(-\infty, +\infty)$.

~~Pf.~~ Proof. Suppose f, g are dependent.
$\Rightarrow C_1 f(t) + C_2 g(t) = 0$ on interval I.
for some values C_1, C_2 not both zero.
Then for some t_0 in I for which $W(f,g)(t_0) \neq 0$

$$\begin{cases} C_1 f(t_0) + C_2 g(t_0) = 0 \\ C_1 f'(t_0) + C_2 g'(t_0) = 0 \end{cases} \qquad \begin{vmatrix} f(t_0) & g(t_0) \\ f'(t_0) & g'(t_0) \end{vmatrix} \neq 0.$$

(as if $a \cdot x = 0$ with $a \neq 0 \Rightarrow x = 0$)

$A\vec{x} = 0$ with $|A| \neq 0 \Rightarrow \vec{x} = (0,0)$

$C_1 = 0 \quad C_2 = 0. \Rightarrow \{f(t), g(t)\}$ lin. indep

Thm 2. (Abel's Theorem).

y_1, y_2 - Solutions to $y'' + p(t)y' + g(t)y = 0$

$p(t), g(t)$ - continuous $\Rightarrow$

$$\boxed{W(y_1, y_2)(t) = Ce^{-\int p(t)\,dt}} \qquad \left(W' + p(t)W = 0\right)$$

if $C = 0 \Rightarrow W = 0$ for all t

if $C \neq 0 \Rightarrow W \neq 0$ for all t.

Thm 3. In the same setup (y_1, y_2 are as above)

either $\{y_1, y_2\}$ - lin. dep. on $I \Leftrightarrow W = 0$ for all t in I

or $\{y_1, y_2\}$ - lin. indep. on $I \Leftrightarrow W \neq 0$ for all t in I

Examples.

1) $t^2 y'' - t(t+2)y' + (t+2)y = 0$ | Standard form: $y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 0$

Find $W(y_1, y_2)$ without solving the equation.

Abel formula: $W(y_1, y_2)(t) = Ce^{-\int p(t)\,dt} =$

$p(t) = -\frac{t+2}{t}$ $\quad e^{-\int p(t)\,dt} = e^{\int \frac{t+2}{t}\,dt} =$

$= e^{\int (1 + \frac{2}{t})\,dt} = e^{t + 2\ln t} = e^t \cdot e^{2\ln t}$

$= e^t \cdot t^2$

$W = Ct^2e^t$, C - constant

2) $\{y_1, y_2\}$ - lin. indep. solutions of $ty'' + 2y' + te^t y = 0$ such that $W(y_1,y_2)(1) = 2$. Find $W(y_1,y_2)(5)$.

3) $y_1 = \cos 4t$, $y_2 = 2\cos 4t$

$\{y_1, y_2\}$ - are these lin. independent?

$$y_2 = 2y_1$$

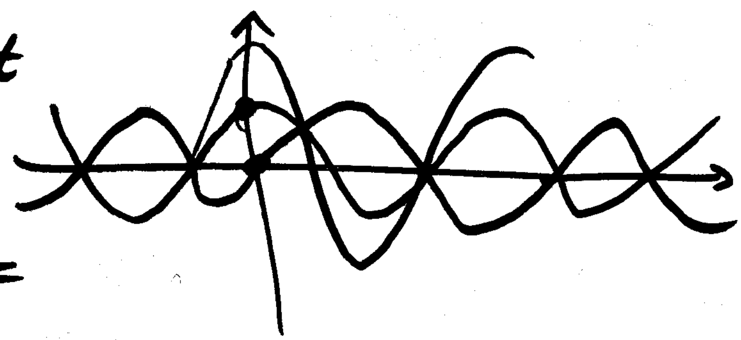
$$2y_1 - y_2 = 0 \leftarrow \text{nontrivial lin. combination}$$

4) $y_1 = \cos^2 x$, $y_2 = 1 + \cos 2x$

$y_1 = \cos^2 x = \dfrac{1+\cos 2x}{2} = \frac{1}{2} y_2 \Rightarrow$ not independent

5) $y_1 = \cos 4t$, $y_2 = \sin 4t$

independent



$$W = \begin{vmatrix} \cos 4t & \sin 4t \\ -4\sin 4t & 4\cos 4t \end{vmatrix} =$$

$$= 4\cos^2 4t + 4\sin^2 4t = 4 \neq 0.$$

$\Rightarrow$ 1) indep.

2) form fund. set if they are both solutions of some ODE.