Math 214.002
Lecture 12.

§3. Second order equations.

§3.1 $ay'' + by' + cy = 0$ (*) $\quad y(t_0) = y_0,\ y'(t_0) = y_0'$

Char. equation:

$ar^2 + br + c = 0$

roots $r_1, r_2 \Rightarrow \begin{array}{l} y_1 = e^{r_1 t} \\ y_2 = e^{r_2 t} \end{array}$ solutions to (*)

$r_1 \neq r_2 \Rightarrow y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ is a solution to (*)

C_1, C_2 are determined by $y(t_0) = y_0, y'(t_0) = y_0'$

Ex. $y'' + 3y' + 2y = 0$

$r^2 + 3r + 2 = 0 \qquad (r+1)(r+2) = 0$

$r_1 = -1, \quad r_2 = -2$

$y_1 = C_1 e^{-t} \quad y_2 = e^{-2t} \Rightarrow \boxed{y = C_1 e^{-t} + C_2 e^{-2t}}$

§3.2. Fundamental Set of Solutions.

All 2nd order linear equations:

$$y'' + p(t)y' + q(t)y = g(t) \quad (1)$$

Thm 1. If $p(t), q(t), g(t)$ are continuous in I, Equation (1) has a unique solution throughout the interval I with any initial data $y(t_0) = y_0, y'(t_0) = y_0'$. ($t_0$ belongs to I)

Ex. $(t^2-4)y'' + (t+2)y' + \ln(2t-1)y = 0.$

$y(1)=0, \ y'(1)=2$

$$y'' + \underbrace{\frac{t+2}{t^2-4}}_{p(t)} y' + \underbrace{\frac{\ln(2t-1)}{t^2-4}}_{q(t)} y = 0$$

Thm 2. y_1, y_2 - two solutions to (1)
then $y = C_1 y_1(t) + C_2 y_2(t)$ is also
a solution for all C_1 and C_2.
(this is only true for linear equations).

Thm 3. $\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$ (2)

Suppose y_1, y_2 - two solutions to (1)
such that $W(y_1,y_2)(t_0) = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)$
$\neq 0$

Then there are constants C_1, C_2 s.t.
$y = C_1 y_1(t) + C_2 y_2(t)$ is the solution to
the IVP (2).

Proof. We know $\underline{y = C_1 y_1 + C_2 y_2}$ solves (1).
To satisfy IVP we need:

$$y(t_0) = C_1 y_1(t_0) + C_2 y_2(t_0) = y_0$$
$$y'(t_0) = C_1 y_1'(t_0) + C_2 y_2'(t_0) = y_0'$$

We need to solve for C_1, C_2 in order to satisfy (2).

Kramer's rule: $C_1 = \dfrac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$

$$C_2 = \frac{\begin{vmatrix} y_1(t_0 & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

(In general, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{cases} a_1 x_1 + b_1 y_1 = c_1 \\ a_2 x_1 + b_2 y_1 = c_2 \end{cases} \Rightarrow$$

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases} \Rightarrow \begin{cases} x = \dfrac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \\ y = \dfrac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \end{cases}$$

Kramer's formula)

If $\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = W(y_1, y_2)(t_0) \neq 0$

formula will provide solution to IVP (2).

$W(y_1, y_2)(t)$ is called Wronskian.

<u>Ex.</u> $y_1 = e^{2t}$ $y_2 = e^{t}$ Calculate $W(y_1, y_2)$

$$W(y_1,y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} =$$

$$= \begin{vmatrix} e^{2t_0} & e^{t_0} \\ 2e^{2t_0} & e^{t_0} \end{vmatrix} = e^{3t_0} - 2e^{3t_0}$$

$$= -e^{3t_0} \neq 0.$$

__Thm 4.__ y_1, y_2 - solutions to (1) such that $W(y_1,y_2)(t_0) \neq 0$ for some point t_0.
Then $y = C_1 y_1(t) + C_2 y_2(t)$ includes all solutions to (1). ∎

$\{y_1, y_2\}$ - __fundamental set__ of solutions.