Math 214002
Lecture 10.

1st order equations : $y' = f(y,t)$

2nd order equations : $\underbrace{p(t)y'' + q(t)y' + r(t)y = g(t)}$

linear $\longrightarrow$

nonlinear : $f(y'', y', y, t) = 0$

§3.1 Homogeneous linear 2nd order DE with const. coefficients.

$g(t) = 0$ $\qquad$ $p(t) = a,\ q(t) = b,\ r(t) = c$

$$\boxed{ay'' + by' + cy = 0} \quad ⊛$$

$y \equiv 0$ const solution of any homogeneous equation.

How do we solve equation ⊛?

1) $a = 0$: special case $\quad by' + cy = 0$

$$\underline{y' + py = 0}, \quad p = \frac{c}{b}$$

$$y' = -py$$

$$y(t) = Ce^{-pt}$$

2) $a \neq 0$: look for solutions in the form

$$y(t) = e^{rt}$$

For which value(s) of r will $y(t) = e^{rt}$ be a solution to ⊛?

Plug $y(t) = e^{rt}$ to ⊛ :

$$\left.\begin{aligned} y(t) &= e^{rt} \\ y'(t) &= re^{rt} \\ y''(t) &= r^2 e^{rt} \end{aligned}\right\} \Rightarrow \begin{aligned} & ay''(t) + by'(t) + cy(t) = 0. \\ & a \cdot r^2 e^{rt} + bre^{rt} + ce^{rt} = 0 \\ & e^{rt}(ar^2 + br + c) = 0 \end{aligned}$$

r - root to this quadratic equation

characteristic equation for DE ⊛: $ay'' + by' + cy = 0$ ← $\boxed{ar^2 + br + c = 0}$

$ar^2 + br + c = 0$ has 2 roots r_1, r_2

So $\underline{ay'' + by' + cy = 0}$ has 2 solutions $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$

Claim:

$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ satisfies ⊛ ⇒ $\boxed{\text{In the case when } r_1 \neq r_2 \text{, real}}$

$$y' = C_1 r_1 e^{r_1 t} + C_2 r_2 e^{r_2 t}$$

$$y'' = C_1 r_1^2 e^{r_1 t} + C_2 r_2^2 e^{r_2 t}$$

$$\begin{aligned} ay'' &= a C_1 r_1^2 e^{r_1 t} + a C_2 r_2^2 e^{r_2 t} \\ + by' &= b C_1 r_1 e^{r_1 t} + b C_2 r_2 e^{r_2 t} \\ + cy &= c C_1 e^{r_1 t} + c C_2 e^{r_2 t} \end{aligned}$$

$$ay'' + by' + cy = C_1 e^{r_1 t}\underbrace{(ar_1^2 + br_1 + c)}_{0} + C_2 e^{r_2 t}\underbrace{(ar_2^2 + br_2 + c)}_{0}$$

$$= 0 \checkmark$$

Moreover, $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ gives all solutions to ⊛ (so-called general solution to ⊛).

Ex. 1 $y'' + 5y' + 6y = 0$

$r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0 \quad r_1 = -2, \ r_2 = -3$

$\boxed{y(t) = C_1 e^{-2t} + C_2 e^{-3t}}$ general solution

To determine C_1, C_2, will need 2 initial conditions $\to \begin{pmatrix} y(0) = y_0 \\ y'(0) = y_0' \end{pmatrix}$ IVP

Suppose $y(0) = 0 \longrightarrow y(0) = \boxed{C_1 + C_2 = 0}$

$y'(0) = 1$

$y'(t) = -2C_1 e^{-2t} - 3C_2 e^{-3t}$

$y'(0) = \boxed{-2C_1 - 3C_2 = 1}$

$C_1 = -C_2$

$2C_2 - 3C_2 = 1 \Rightarrow -C_2 = 1$

$\boxed{C_1 = 1 \Leftarrow C_2 = -1}$

Solution to IVP:

$\boxed{y(t) = e^{-2t} - e^{-3t}}$

Ex. 2 $y'' - 3y' + 2y = 0 \quad y(0) = 2, \ y'(0) = 1$

$r^2 - 3r + 2 = 0 \quad (r-2)(r-1) = 0$

$r_1 = 1, \ r_2 = 2$

Gen. sol. : $y(t) = C_1 e^t + C_2 e^{2t}$

If we solve for C_1, C_2 from $y(0) = 2, y'(0) = 1$, we will get solution to the IVP:

$y(0) = C_1 + C_2 = 2$

$y'(0) = C_1 + 2C_2 = 1$

$\Rightarrow \left| \begin{array}{l} 1 - C_2 = 2 \\ C_1 = 1 - 2C_2 \end{array} \right.$

$\boxed{C_2 = -1, \ C_1 = 3} \Rightarrow \boxed{y(t) = 3e^t - e^{2t}}$

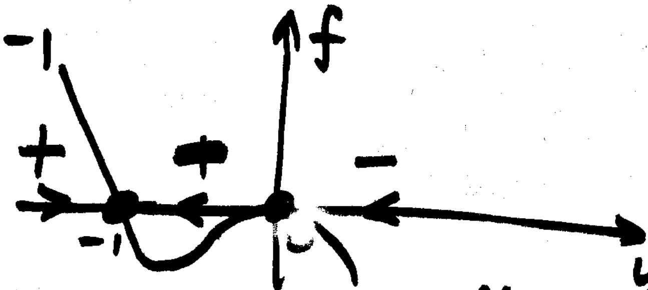
$$y' = f(y)$$

$$f(y) = -(y+1)y^2 = 0 \longrightarrow \begin{array}{l} y = 0 \\ y+1 = 0 \end{array}$$

1) find zeros: $y = 0$

$y = -1$

2) graph $f(y)$:



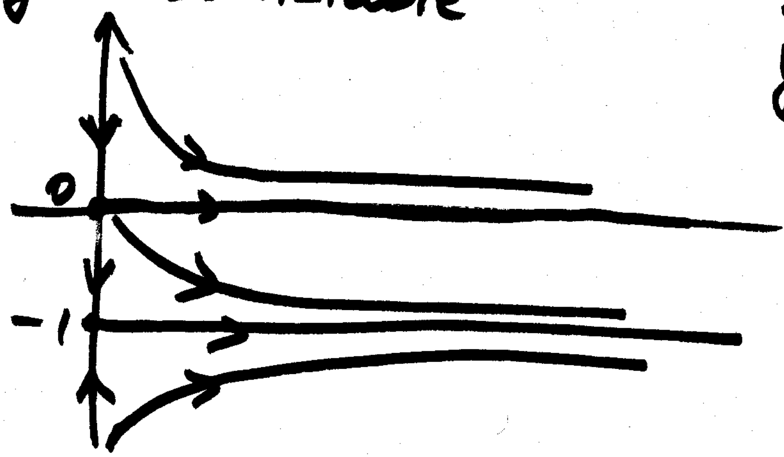
$y = -1$ stable

$y = 0$ semistable

$y = 1 \quad f(y) = -$

$y = -\frac{1}{2} \quad f(y) = -$

$y = -2 \quad f(y) = +$



Quiz prep:

1) Linear/nonlinear equations, existence of Solution.

Thm 1: $y' + p(t)y = g(t)$, if $p(t), g(t)$ - cts in an interval $\Rightarrow$ $\exists!$ sol.

Thm 2: $y' = f(t,y)$ if $f, \frac{\partial f}{\partial y}$ - continuous in a region $\Rightarrow$ exists a unique solution.

ex.1: $y' = \frac{1}{t+y}$

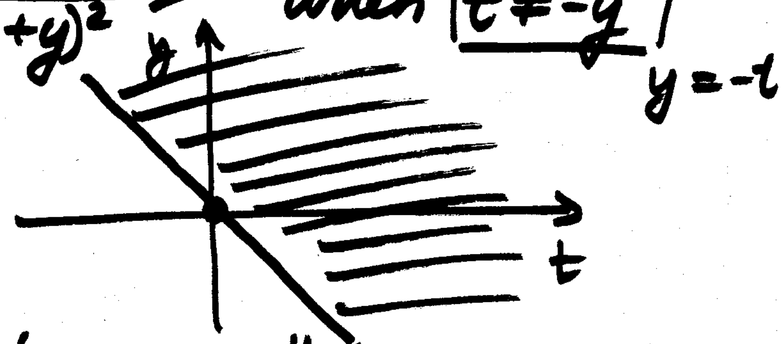
find an ~~interval~~ region in ty-plane where there exists a unique solution

$f(t,y) = \frac{1}{t+y} = (t+y)^{-1}$ continuous when $\boxed{t \neq -y}$

$\frac{\partial f}{\partial y}(t,y) = -\frac{1}{(t+y)^2}$

either $y+t<0$

or $y+t>0$



if $y(2)=5 \rightarrow$ pick Second region where $\boxed{y+t>0}$

2) Autonomous equations.

ex.2 $y' = -(y+1)y^2$

→ Classify all equilibria

→ draw phase portrait