NAME : ______________________

MATH 251
Summer 2003
Final Exam
Take home

ID : ______________________

INSTRUCTOR : ______________________

This exam is due at the **beginning** of class on Thursday the 7^{th} of August, 2003. Class attendance on that day is compulsory. When you hand in this exam, please read the statement below and sign this page, returning it with your exam to your instructor. Failure to do this will result in a deduction of your final exam score.

This is a take-home final exam. You may use your notes and any book(s). However, you may not receive any from any other person, except perhaps from myself (a clarification of a problem). By signing this you signify that you have read the above and have not breached the above conditions for this exam, if you feel you cannot fulfil these requirements please see me.

Your signature: ______________________

PLEASE DO NOT WRITE IN THE BOX BELOW.

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Total: __________

1. (15 points)
   (a) Give an example of a first order autonomous differential equation that has exactly 3 equilibrium solutions.
   (b) Find a second order linear equation whose general solution is $y(t) = C_1 e^{-2t} + C_2 t e^{-2t} + \cos t$.
   (c) Find an exact equation whose general solution is $x^2 y + x e^y = C$. Verify the exactness of your answer.

2. (18 points) Consider the initial value problem

$$y' + y = te^t, \quad y(0) = 1.$$

   (a) Solve it using the method of integrating factor.
   (b) Solve it using the Laplace transform.

3. (15 points) The velocity of a boat moving in a lake satisfies the equation

$$v' = 900 - 4v^2.$$

   (a) Find and classify the equilibrium solutions. Justify your answer.
   (b) Solve this equation. You may leave you answer in implicit form.

4. (18 points) A mass of $1\,kg$ stretches a spring $1\,m$. The system has damping $7\,\frac{N\cdot s}{m}$. At $t = 0$, the mass moves away from its equilibrium position with initial velocity $2\,\frac{m}{s}$. In addition, at $t = 0$ an external force $F(t) = \cos t$ is applied to the mass, but it is discontinued at $t = 3\pi$. Assume gravity $g = 10\,\frac{m}{s^2}$.
   (a) Write an initial value problem that models this system.
   (b) Solve the initial value problem.
   (c) What is the position of the mass at $t = \pi$? At $t = 4\pi$?
   (d) In the absence of damping (letting $\gamma = 0$) what would the system's natural frequency ω_0 be?

5. (15 points)
   (a) Solve the initial value problem:

$$X' = \begin{bmatrix} 2 & 5 \\ -3 & -6 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} -4 \\ 2 \end{bmatrix}.$$

   (b) Classify the type and stability of the critical point at $(0,\,0)$.

6. (15 points) Consider the system

$$x' = (x+y)(x-y),$$
$$y' = (x-2)(y+1).$$

   (a) Find all critical points of the system (there are 4).
   (b) Find the linearized matrix of the system.
   (c) Linearize the system about any 2 of the 4 critical points, and determine the type and stability of each.

7. (12 points)
   (a) For what positive values of λ does the boundary value problem

$$x'' + \lambda x = 0, \qquad x'(0) = 0, \qquad x(\pi) = 0$$

   have a nontrivial solution?
   (b) What are the corresponding eigenfunctions?

8. (15 points) Let

$$f(x) = \begin{cases} 1-x, & 0 < x < 2 \\ -1, & 2 \leq x < 3 \end{cases}$$

   (a) Sketch the even, period 6 extension of $f(x)$ on the interval $[-9,\, 9]$.
   (b) Sketch the odd, period 6 extension of $f(x)$ on the interval $[-9,\, 9]$.
   (c) Set up, but do not evaluate, the integral(s) that will give the Fourier sine coefficients of the odd extension.
   (d) To what value does the sine series above converge to at $x = -3$, $x = 0$, and $x = 2$?

9. (12 points) Consider the boundary value problem

$$t^2 u_{tt} = x u_{xt}$$
$$u(0,t) = u(1,t) = 0$$

   (a) By setting $u(x,\, t) = X(x)\, T(t)$, separate the equation into 2 ordinary differential equations.
   (b) What new boundary condition must the equation of $X(x)$ satisfy?

10. (15 points) Solve the nonhomogeneous heat equation

$$10u_{xx} = u_t, \qquad 0 < x < 6, \quad t > 0$$
$$u(0,t) = 20, \quad u(6,t) = 80, \qquad t > 0$$
$$u(x,0) = 20 + 10x + 5\sin \pi x - 10 \sin \frac{3\pi x}{2}$$