NAME : ______________________

MATH 251
Spring 2003
Final Exam
May 8, 2003

ID : ______________________

INSTRUCTOR : ______________________

There are **15** questions on **5** pages. Please read each problem carefully before starting to solve it. For each multiple choice problem 4 answers are given, only one of which is correct. Mark only one choice. For partial credit questions, all work must be shown - **credit will not be given for an answer unsupported by work.**

NO CALCULATORS ARE ALLOWED.
PLEASE DO NOT WRITE IN THE BOX BELOW.

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1. (6 points) Consider the undamped system of a $7kg$ mass hanging from a spring. An external force of $3\sin(10t)$ newtons is applied to the system, which then enters resonance. What is the spring constant?
   (a) 7
   (b) 10
   (c) 100
   (d) 700

2. (6 points) What is the inverse Laplace transform of $F(s) = e^{-3s}\frac{1}{s+2}$?
   (a) $u_3(t) + \frac{1}{2}\delta(t-3)$
   (b) $u_3(t)e^{2t+3}$
   (c) $u_3(t)e^{-2t+6}$
   (d) $\delta(t-3)e^{-2t}$

3. (6 points) What is the partial fraction expansion of $\frac{7s-2}{s^2(s-2)}$?
   (a) $\frac{-3s+1}{s^2} + \frac{1}{s-2}$
   (b) $\frac{1}{s} + \frac{3}{s^2} + \frac{1}{s-2}$
   (c) $\frac{-3}{s} + \frac{1}{s^2} + \frac{3}{s-2}$
   (d) $\frac{1}{s^2} + \frac{1}{s-2}$

4. (6 points) Consider the following two differential equations:

$$\begin{array}{llll} \text{I} & y'' + ay' + by = 0 & y(0) = 0 & y'(0) = 2 \\ \text{II} & y'' + ay' + by = 0 & y(0) = 0 & y(\pi) = 2 \end{array}$$

   Where $a, b \in \mathbb{R}$ Which of the following statements are true?
   (a) Both I and II always have a unique solution on some interval.
   (b) Only I always has a solution on some interval.
   (c) Only II always has a solution on some interval.
   (d) None of the above.

5. (6 points) What is the stability of the equilibrium solution $y(t) = 3$ for the following autonomous differential equation?

$$y' = y(y-3)(y+3)$$

   (a) Stable.
   (b) Unstable.

(c) Semi-stable.

(d) None of the above.

6. (6 points) Which of the following equations is exact?

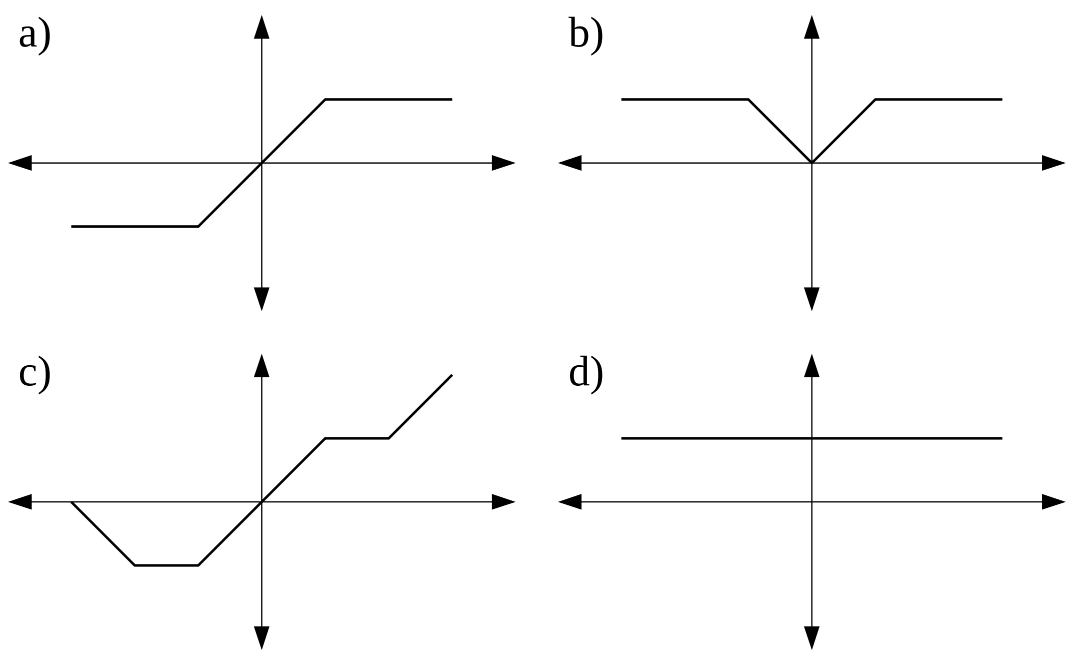
   (a) $te^{yt}\frac{dy}{dt} + ye^{yt} + 2t = 0$

   (b) $(ye^{yt} + 2t)\frac{dy}{dt} = te^{yt}$

   (c) $(ye^{yt} + 2t)\frac{dy}{dt} + te^{yt} = 0$

   (d) $te^{yt}\frac{dy}{dt} + ye^{yt} - 2t = 0$

7. (6 points) Which of the following graphs would have a fourier series consisting only of sine terms? Note: only one period of each function is shown.




8. (6 points) Which of the following pair of functions are not linearly independant?

   (a) $y_1(t) = t$ and $y_2(t) = 1$.

   (b) $y_1(t) = \sin 2t$ and $y_2(t) = \cos 2t$.

   (c) $y_1(t) = e^{3t}$ and $y_2(t) = e^{3t-2}$.

   (d) $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$.

9. (16 points)

(a) What is the interval on which a solution the following differential equation is certain to exist?

$$t^2y'' + ty' - y = 0 \qquad y(1) = 2 \qquad y'(1) = 0$$

(b) Given $y(t) = t$ is a solution to the above differential equation (you need not check this) what is the general solution?

10. (14 points) Consider the following nonlinear system.

$$\begin{aligned} x' &= xy - 3x \\ y' &= 2xy + 2y \end{aligned}$$

(a) Find the linearized matrix for this system.

(b) Find the critical points.

(c) Chose one of the critical points and state what is its type and stability.

11. (14 points) Consider the following function, defined on the interval (0,2).

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2 & 1 \leqslant x < 2 \end{cases}$$

(a) Graph the **even**, period 4, extension of $f(x)$ on $(-4, 4)$.

(b) Graph the **odd**, period 4, extension of $f(x)$ on $(-4, 4)$.

(c) Which of the above two has a cosine series, and which has a sine series?

(d) What does the fourier series representing part b (the odd extension) converge to at $x = -2$, $x = \frac{1}{2}$ and $x = 3$?

12. (12 points) Consider the following partial differential equation

$$u_{yy} + 2u_y = y^2 u_{xx}$$

(a) Separate this equation into two ordinary differential equations.

(b) Translate the following boundary conditions on the above partial differential equation to conditions on the ordinary differential equations found above.

$$u(0, y) = 0 \qquad u(L, y) = 0$$

13. (14 points) For the following boundary problem find all **positive** eigenvalues and their corresponding eigenfunctions.

$$X'' + \lambda X = 0 \qquad X(0) = 0 \qquad X'(\pi) = 0$$

14. (14 points) Consider a thin rod of length $10cm$ with thermal diffusivity $\alpha^2 = 2\frac{cm^2}{s}$ and perfectly insulated ends. Using the variable $x$ as the distance from the left end of the rod the initial temperature of this rod is

$$f(x) = 3 + \cos\frac{\pi x}{5} - 5\cos\frac{\pi}{2}$$

   (a) Construct the partial differential equation for this situation, also give boundary conditions.

   (b) Solve this partial differential equation.

   (c) Determine the steady-state temperature.

15. (18 points)

   (a) Find the general solution to the following system.

$$X'(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} X$$

   (b) Find the specific solution which meets the following initial condition.

$$X(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$