Intersections of Descending Sequences of Affinely Equivalent Convex Bodies

Jim Lawrence
Department of Mathematical Sciences
George Mason University, Fairfax, VA 22030

Abstract. *Here it is shown that a set in Euclidean space can be represented as the intersection of a descending sequence of sets affinely equivalent to a given convex body, or arbitrarily closely approximated from above by sets affinely equivalent to the body, if and only if it is affinely equivalent to an affine retract of the body. For the special case in which the body is a simplex, the statement concerning descending sequences is a well-known result of Borovikov.*

1. Introduction. Answering a question of Kolmogorov, Borovikov [1] showed that the intersection of a descending sequence of simplexes in $\mathbb{R}^d$ is a simplex. In this paper, given a compact, convex set K lying in $\mathbb{R}^d$, we characterize the possible intersections of descending sequences of sets affinely equivalent to K, thereby obtaining a generalization of the result of Borovikov. This characterization is obtained as a consequence of a similar characterization of the family of sets that can be arbitrarily closely approximated from above by sets affinely equivalent to K. It is shown that these families coincide (Theorems 3 and 4), and consist of those compact convex sets that are affinely equivalent to affine retracts of K. In the presence of the characterizations of simplexes by Choquet [2] and Rogers, Shephard [6], this can be seen to generalize the Borovikov result: It is not difficult to show that any affine retract of a Choquet simplex is itself a Choquet simplex, a result first noticed by Semadeni [7]. For completeness, Semadeni's result with a proof is given below in Theorem 5. For a concise proof of the Choquet-Rogers-Shephard result, see Martini's paper [4]. The theorem of Borovikov and Kolmogorov was extended by Eggleston, Grünbaum, Klee [3] to include intersections of chains of compact Choquet simplexes in topological linear spaces. See also Phelps [5]. For an informative survey of work on Choquet simplexes up to 2004, see Soltan [9].

Some questions concerning special cases of these results are briefly considered.

2. Affine retracts. A function $\alpha : \mathbb{R}^d \to \mathbb{R}^d$ is *affine* provided that it is of the form $\alpha(x) = \lambda(x) + b$, where $\lambda : \mathbb{R}^d \to \mathbb{R}^d$ is linear and $b \in \mathbb{R}^d$. Such an affine function α is termed *nonsingular* if it has an inverse $\alpha^{-1} : \mathbb{R}^d \to \mathbb{R}^d$, necessarily also an affine function.

Sets X and Y in $\mathbb{R}^d$ are termed *affinely equivalent* if there is a nonsingular affine mapping $\alpha : \mathbb{R}^d \to \mathbb{R}^d$ such that $Y = \alpha(X)$. An *affine retraction* of a compact, convex set $K \subseteq \mathbb{R}^d$ is an affine function $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ such that $\varphi(\varphi(x)) = \varphi(x)$ for each $x \in \mathbb{R}^d$, that maps K into itself. An *affine retract* of such a set $K \subseteq \mathbb{R}^d$ is the image $\varphi(K)$ of K under an affine retraction. Any affine retract of K is a subset of K.

The first theorem gives some characterizations of affine retracts. Some necessary lemmas, that also serve to place the idea of affine retract into a suitable general framework, precede this result.

Lemma 1. *Suppose X is a set and $f : X \to X$ is a function mapping X into itself. Then there is a unique maximal subset $W \subseteq X$ such that $f(W) \supseteq W$. Furthermore,*
(a) $W = \{y \in X :$ there is a sequence $w_1, w_2, \ldots$ of elements of X such that $y = f(w_1)$ and $w_{k-1} = f(w_k)$ when $k \geq 2\}$;
(b) $W \subseteq \bigcap_{k=1}^{\infty} f^{(k)}(X)$; and
(c) $f(W) = W$, and W is the unique maximal subset of X satisfying this equality.

Proof. If each element of a collection of subsets Y of X has the property that $f(Y) \supseteq Y$, then the union of the sets in the collection also has this property. The collection of such subsets of X is nonempty, since $\emptyset$ is such a set. Letting W denote the union of all such sets, it is clear that W is the unique maximal subset of X such that $f(W) \supseteq W$.

Suppose $y \in X$ and there exists a sequence $w_1, w_2, \ldots$ as in (a). Letting $Y = \{y, w_1, w_2, \ldots\}$, it is clear that $f(Y) \supseteq Y$, so $y \in Y \subseteq W$. Suppose now that $y \in f(W)$. Let $w_1 \in W$ be such that $y = f(w_1)$. Since $w_1 \in W \subseteq f(W)$, there is $w_2 \in W$ such that $f(w_2) = w_1$. Continuing in this way, by induction we obtain a sequence $w_1, w_2, \ldots$ such that $f(w_1) = y$ and $f(w_k) = w_{k-1}$ when $k > 1$. We see that (a) holds.

Since $X \supseteq W$, we have that $f(X) \supseteq f(W) \supseteq W$. Inductively, $f^{(k)}(X) \supseteq W$. Therefore, $W \subseteq \bigcap_{k=1}^{\infty} f^{(k)}(X)$.

For (c), let $V = f(W)$ and note that from $f(W) \supseteq W$ it follows that $f(V) \supseteq V$. Since W is the maximal set having this property, $V \subseteq W$. Then $f(W) = W$. Uniqueness follows at once. $\square$

LEMMA 2. *If, in the previous lemma, the set X has the structure of a compact Hausdorff space and the function f is continuous, then $W = \bigcap_{k=1}^{\infty} f^{(k)}(X)$.*

Proof. In view of (b) of Lemma 1, we need only show that $\bigcap_k f^{(k)}(X) \subseteq W$. Suppose $y \in \bigcap_k f^{(k)}(X)$. For $k = 1, 2, \ldots$, let $A_k = \{w \in X : f^{(k)}(w) = y\}$. By assumption, $A_k \neq \emptyset$; also the sets A_k are closed subsets of X and therefore compact. For $k = 1, 2, 3, \ldots$, let $B_k = f^{(k)}(A_{k+1})$. Then each B_k is compact and nonempty. Furthermore, $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$. It follows that $\bigcap_k B_k \neq \emptyset$. Let w_1 be an element of this set. Then $f(w_1) = y$ and $w_1 \in \bigcap_k f^{(k)}(X)$. Inductively, choose $w_2, w_3, \ldots$ such that, for each $k = 2, 3, \ldots$, $f(w_k) = w_{k-1}$ and $w_k \in \bigcap_k f^{(k)}(X)$. Then $y, w_1, w_2, \ldots$ are as in (a) of Lemma 1. $\square$

LEMMA 3. *Let $X \subseteq \mathbb{R}^d$ be a compact, convex set. Suppose that $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is an affine function mapping X to a subset Y and that $\phi(Y) = Y$. Then Y is an affine retract of X.*

Proof. If A denotes the affine space spanned by Y, then the restriction of ϕ to A is invertible; there is a function $\tau : A \to A$ such that $\tau(\phi(x)) = x$ for each $x \in A$. Letting ρ be the composition, $\rho = \tau \circ \phi$, then ρ is an affine retraction of X to Y, and Y is an affine retract of X. $\square$

THEOREM 1. *Suppose X is a compact, convex set in $\mathbb{R}^d$ and $Y \subseteq X$. Then the following statements are equivalent.*
(a) There is an affine function $\alpha : X \to X$ such that Y is the maximal subset of X for which $\alpha(Y) = Y$.
(b) There is a nonsingular affine function $\alpha : X \to X$ such that Y is the maximal subset of X for which $\alpha(Y) = Y$.
(c) There is an affine function $\alpha : X \to X$ such that $Y = \bigcap_k \alpha^{(k)}(X)$.
(d) There is a nonsingular affine function $\alpha : X \to X$ such that $Y = \bigcap_k \alpha^{(k)}(X)$.
(e) The set Y is an affine retract of X.

Proof. Clearly (b) implies (a) and (d) implies (c). Lemmas 1 and 2 show that (a) and (c) are equivalent, and that (b) and (d) are equivalent.

Suppose that (e) holds; we verify (d). Let $\beta : X \to Y$ be an affine retraction. Define α by $\alpha(x) = \frac{1}{2}(x + \beta(x))$. Since β is an affine retraction, $\beta(\alpha(x)) = \beta(x)$, so that $x = 2\alpha(x) - \beta(\alpha(x))$. It follows that α is nonsingular, its inverse being given by $u \mapsto 2u - \beta(u)$. For $x \in Y$, $\alpha(x) = \beta(x) = x$, so $Y \subseteq \alpha(Y)$. Suppose $x \in X \setminus Y$. We must show that $x \notin \bigcap_k \alpha^{(k)}(X)$. Let ϵ denote the distance of x to Y. For each k, $\alpha^{(k)}(X) \subseteq \frac{1}{2^k}X + \frac{2^k-1}{2^k}Y$. Since X is compact, $\frac{1}{2^k}X$ is contained in an open ball of radius ϵ centered at the origin, when k is large. Therefore, for such k, the distance of x to $\alpha^{(k)}(X)$ is positive, so that $x \notin \bigcap_k \alpha^{(k)}(X)$, and (d) holds.

Suppose (c) holds; we verify (e). Let α be as in (c). The sequence $\alpha^{(k)}$ must have a subsequence $\alpha^{(k_j)}$ that converges pointwise to a (necessarily affine) function $\beta : \mathbb{R}^d \to \mathbb{R}^d$. Clearly $\beta(X) = Y = \beta(Y)$. By Lemma 3, (e) holds. $\square$

When the set X of the theorem is the simplex of probability measures on a finite set, the computation of the set W has been studied by Sierksma [8]; the theorem is of some interest when dealing with Markov chains.

3. A general result. Part of the proof of the main theorems will be based upon the following general result, in which we are given a compact metric space X together with two collections of functions $\mathcal{F}$, $\mathcal{G}$ mapping X to itself. The following assumptions are made.

(a) The identity function on X is in $\mathcal{F}$; $\mathcal{G} \subseteq \mathcal{F}$.
(b) If $f_n \in \mathcal{F}$ $(n = 1, 2, \dots)$ and the sequence $\{f_n\}$ converges to f uniformly, then $f \in \mathcal{F}$.
(c) Any sequence $\{f_n\}$ in $\mathcal{F}$ has a uniformly convergent subsequence.
(d) The collections $\mathcal{F}$ and $\mathcal{G}$ are closed under composition.
(e) If $g \in \mathcal{G}$, $f \in \mathcal{F}$, and $g(X) \supseteq f(X)$, then there is $h \in \mathcal{F}$ such that the composition $g \circ h = f$.
(f) The functions in $\mathcal{G}$ are injective.

The following theorem is applied in the next section. In this application, the set X is a compact, convex set in $\mathbb{R}^d$, $\mathcal{F}$ is the set of all affine linear functions $f : \mathbb{R}^d \to \mathbb{R}^d$ such that $f(X) \subseteq X$, and $\mathcal{G}$ is the set of nonsingular affine functions that are in $\mathcal{F}$. For the function h in (e), we may then take $h = g^{-1} \circ f$.

Theorem 2. *Suppose that X, $\mathcal{F}$, and $\mathcal{G}$ are as above. Suppose also that $Y \subseteq X$, and Y can be approximated arbitrarily well (with respect to the*

Hausdoff metric) by sets of the form $f(X)$, where $f \in \mathcal{G}$ and $f(X) \supseteq Y$. Then there is a set $Z \subseteq X$, a function $h \in \mathcal{F}$ such that $h \circ h = h$ with $h(X) = Z$, and a function $g \in \mathcal{F}$ such that the restriction of g to Z is a bijection, $g : Z \to Y$.

Proof. Let D denote the metric on X.

For $n = 1, 2, \ldots$, let $g_n : X \to X$ be an element of $\mathcal{G}$ such that $Y \subseteq g_n(X)$ and the Hausdorff distance between $g_n(X)$ and Y is at most $\frac{1}{n}$. In view of (b), by passing to a subsequence if necessary, we may assume that $\{g_n\}$ converges uniformly to a function $g \in \mathcal{F}$. Clearly $g(X) \subseteq Y$, and by uniform convergence, $g(X) = Y$.

Considering (e), for each n we may find $h_n \in \mathcal{F}$ such that $g_n \circ h_n = g$. Again passing to a subsequence, we may assume that $h_n \to h \in \mathcal{F}$ uniformly.

We show that $g(h(x)) = g(x)$ for each $x \in X$. Suppose $\epsilon > 0$. Choose $\delta > 0$ such that if $D(u, v) < \delta$ then $D(g(u), g(v)) < \frac{\epsilon}{2}$. Choose a positive integer N such that if $n > N$ then $D(h(x), h_n(x)) < \delta$ and $D(g(x), g_n(x)) < \frac{\epsilon}{2}$. Then $D(g(h(x)), g(x)) = D(g(h(x)), g_n(h_n(x))) < D(g(h(x)), g(h_n(x))) + D(g(h_n(x)), g_n(h_n(x))) < \epsilon$ when $n > N$. Since this holds for each $\epsilon > 0$, we have $g(h(x)) = g(x)$.

We show that $h \circ h = h$. Let $y = h(x)$. For each n we have $g_n(h_n(y)) = g(y) = g(x) = g_n(h_n(x))$. Since $g_n \in \mathcal{G}$, g_n is injective, so $h_n(y) = h_n(x)$. Taking the limit, we have $h(h(x)) = h(y) = h(x)$.

Finally, we show that g maps $h(X)$ bijectively to Y. If $y = g(x) \in g(X)$ then, as we have seen, $g(h(x)) = g(x)$, so g maps $h(X)$ onto Y. Suppose $x_1, x_2 \in X$, so that $h(x_1), h(x_2) \in h(X)$, and suppose $g(h(x_1)) = g(h(x_2))$. We must show that $h(x_1) = h(x_2)$. But $g_n(h_n(x_1)) = g(x_1) = g(h(x_1))$ and $g_n(h_n(x_2)) = g(x_2) = g(h(x_2))$ for each n, so by injectivity of the g_n's we have $h_n(x_1) = h_n(x_2)$ for each n, and upon taking limits we obtain the desired equality.

The result holds, with $Z = h(X)$. □

4. Approximation from above and intersections of sequences. Let U^d denote the interior of the unit ball centered at the origin in $\mathbb{R}^d$. We say that Y *can be approximated arbitrarily well from above by compact, convex sets affinely equivalent to* X provided that, for each $\epsilon > 0$, there exists a compact convex set X' affinely equivalent to X such that $Y \subseteq X' \subseteq Y + \epsilon U^d$.

In this definition, the set U^d can be replaced by any nonempty bounded open convex set containing the origin without changing the meaning.

THEOREM 3. *Let X and Y be compact, convex sets. Then Y can be approximated arbitrarily well from above by compact, convex sets affinely equivalent to X if and only if Y is affinely equivalent to an affine retract of X.*

Proof. First suppose that Y is an affine retract of X. By Theorem 1, there is a nonsingular affine function α which maps X to itself and for which $Y = \bigcap_k \alpha^{(k)}(X)$. If $\epsilon > 0$, the sets $\alpha^{(k)}(X) \setminus (Y + \epsilon U^d)$ form a decreasing sequence of compact sets with empty intersection; therefore, for some $\bar{k}$, this set is empty. Then $\alpha^{(\bar{k})}(X)$ contains Y and is contained in $Y + \epsilon U^d$. The sets $\alpha^{(k)}(X)$ are affinely equivalent to X, so it follows that Y can be approximated arbitrarily well from above by sets affinely equivalent to X.

The property of being representable as such an intersection is invariant under affine equivalence, so the "converse" part is done.

For the other direction, we need only observe that, for $\mathcal{G}$ the set of nonsingular affine functions mapping X into itself and $\mathcal{F}$ the set of all affine functions mapping X into itself, the hypotheses of Theorem 2 are satisfied, and that theorem yields the desired result. $\square$

THEOREM 4. *Let $X \subseteq \mathbb{R}^d$ be a compact, convex set. A convex set Y can be represented as the intersection of a chain of sets affinely equivalent to X if and only if Y is affinely equivalent to an affine retract of X.*

Proof. If Y is an affine retract of X then, by Theorem 1, it can be represented as the intersection of a chain of sets affinely equivalent to X.

Suppose $X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$, where each set X_k is affinely equivalent to X, and $Y = \bigcap X_k$. If $\epsilon > 0$, the sets $X_k \setminus (Y + \epsilon U^d)$ form a decreasing sequence of compact sets with empty intersection; therefore, for some $\bar{k}$, this set is empty. Then $X_{\bar{k}}$ contains Y and is contained in $Y + \epsilon U^d$. The sets X_k are affinely equivalent to X, so it follows that Y can be approximated arbitrarily well by sets affinely equivalent to X which contain it. By Theorem 3, Y is an affine retract of X. $\square$

THEOREM 5. (Semadeni [7].) *Suppose that $T \subseteq \mathbb{R}^d$ is a simplex and $\pi : \mathbb{R}^d \to \mathbb{R}^d$ is an affine retraction of T. Then $\pi(T)$ is a simplex.*

Proof. We use the Choquet-Rogers-Shephard theorem: The compact set S is a simplex if and only if, for each $t \in \mathbb{R}^d$, $(S+t) \cap S$ is either empty, a singleton, or homothetic to S.

Suppose $t \in \mathbb{R}^d$. If $t \notin \pi(\mathbb{R}^d)$, then $(\pi(T)+t) \cap \pi(T) = \emptyset$. If $t \in \mathbb{R}^d$, then $\pi(t) = t$ and $(\pi(T)+t) \cap \pi(T) = \pi(T+t) \cap \pi(T) = \pi((T+t) \cap T)$. If $(T+t) \cap T$ is a singleton or empty, so is $\pi((T+t) \cap T)$. If $(T+t) \cap T = \lambda T + s$, then $\pi((T+t) \cap T) = \lambda\pi(T) + \pi(s)$. □

THEOREM 6. (Borovikov [1].) *The intersection of a descending sequence of simplexes in $\mathbb{R}^d$ is a simplex.*

Proof. This is immediate from Theorems 4 and 5. □

5. Best approximations. In this section we briefly consider a question that immediately presents itself given the results above: If Y is not affinely equivalent to an affine retract of X, how far must a set affinely equivalent to X that contains Y be, from Y? There are many ways to measure this. In Theorem 7, we consider the volume in the affine space spanned by Y of an affine projection of X. Also, in Theorem 8, we show that for any convex polytope P the set of possible dimensions of simplexes containing P but properly containing no other such simplex is bounded above.

The following lemma will be of use.

LEMMA 4. *Suppose X is a compact, convex set in $\mathbb{R}^d$ and $\epsilon > 0$. There is a number ν strictly larger than the d-measure of X such that, if Y is a compact, convex set in $\mathbb{R}^d$ whose d-measure is less than ν and $Y \supseteq X$, then $Y \subseteq X + \epsilon U^d$.*

Proof. Let $W \subseteq \mathbb{R}^d$ consist of all points at distance ϵ from X. Then W is a nonempty compact set missing X. The function $f(y) = \mathrm{vol}(\mathrm{conv}(\{y\} \cup X$ is continuous and its value is larger than $\mathrm{vol}(X)$ at each point of W. Therefore the minimum value of f on W exists and is larger than $\mathrm{vol}(X)$. Let ν be this value; then if $Y \supseteq X$ and $\mathrm{vol}(Y) < \nu$, $Y \subseteq X + \epsilon U^d$. □

We denote the number ν in the lemma by $\nu(X, \epsilon)$.

THEOREM 7. *Suppose X and Y are compact, convex sets in $\mathbb{R}^d$, and $\dim Y \leq \dim X$. Let A be the affine space spanned by Y. Let ν be the volume of Y in this space. If Y is not affinely equivalent to an affine retract of X, then there is a number $\mu > \nu$ such that, for any polytope X' affinely* *equivalent to X with $Y \subseteq X'$, the volume in A of any affine projection of X' to A is at least μ.*

Proof. Suppose that for each $\mu > \nu$ there is a compact, convex set X' affinely equivalent to X such that $X' \supseteq Y$ and $\mathrm{vol}(X') < \mu$. Let ϵ be a positive real number. Let $\mu = \nu(Y, \epsilon) > \nu$. If X' is as above then by Lemma 4, $\pi_A(X') \subseteq Y + \epsilon U^d$, where π_A denotes orthogonal projection. Letting b denote the maximum value of $|x - \pi_A(x)|$ on X', the function $x \mapsto \frac{\epsilon}{b}x + (1 - \frac{\epsilon}{b})\pi_A(x)$ is a nonsingular affine mapping that takes X' to a set X'' contained in $Y + \epsilon U^d$; so Y can be approximated arbitrarily well by compact, convex sets affinely equivalent to X. Then by Theorem 5, Y is affinely equivalent to an affine retract of X. $\square$

THEOREM 8. *Suppose P is a convex polytope having n vertices, T is a simplex that contains P, and there exists no simplex $T' \neq T$ such that $P \subseteq T' \subseteq T$. Then $\dim(T) < \binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

Proof. Without loss of generality we may assume that T lies in the vector space $\mathbb{R}^X$ of real-valued functions on the finite set X, and that $T = \{f \in \mathbb{R}^X : f(x) \geq 0 \text{ for each } x \in X \text{ and } \sum_{x \in X} f(x) = 1\}$. Let the vertices of P be $v_1, v_2, \ldots, v_n \in \mathbb{R}^X$. For $f \in \mathbb{R}^X$ let the support of f be denoted by $\mathrm{supp}(f)$, so that $\mathrm{supp}(f) = \{x \in X : f(x) \neq 0\}$. For any pair $x, y \in X$ with $x \neq y$, and for ϵ such that $0 < \epsilon \leq 1$, the set $\{f \in T : f(x) - \epsilon f(y) \geq 0\}$ is a simplex properly contained in T. It follows from the hypotheses that, for any pair x and y of distinct elements of X, there exists a vertex v of P such that $v(x) - \epsilon v(y) < 0$, for each $\epsilon > 0$. In order for this to be the case we must have $v(x) = 0$ and $v(y) > 0$; that is, $x \notin \mathrm{supp}(v)$ and $y \in \mathrm{supp}(v)$. It follows that, if we put $S(x) = \{i : x \in \mathrm{supp}(v_i)\} \subseteq \{1, 2, \ldots, n\}$, then $x \neq y$ implies $S(y) \not\subseteq S(x)$. The sets $S(x)$ are distinct and no one contains another. By a theorem of Sperner, there can be at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ of them. $\square$

Probably the simplest nontrivial situation for each of the two theorems above is that of a square. If, in Theorem 7, Y is a square, then we may take $\mu = 2\nu$; and if in Theorem 8, P is a square, then the dimension of T is bounded by 3.

6. Notes and acknowledgements. The paper of Eggleston, Grünbaum, and Klee [3] contains discussions of problems in the same spirit as those of the previous sections. It also presented another possible proof of Borovikov's result (Theorem [1], based upon the assumption of an affirmative answer to the following question.

If S is a d-simplex and P is an r-polytope in S having at most $r+2$ vertices, must S have an r-face whose r-measure is at least that of P?

For $r = d-1$, the question was answered affirmatively in the paper, and it was noted that the answer is also positive when $r = 0$, 1, or d. In other cases the question is apparently still unresolved. The question without the restriction on the number of vertices of P was also raised. Walkup [10] showed by example that the answer is negative when the restriction on P is dropped.

It would be nice to have results analogous to the foregoing, for projectively equivalent compact, convex sets.

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