The Odd-Even Invariant for Graphs

by

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The odd-even invariant for graphs is the graphic version of the odd-even invariant for oriented matroids. Here, simple properties of this invariant are verified, and for certain graphs, including chordal graphs and complete bipartite graphs, its value is determined. The odd-even chromatic polynomial is introduced, its coefficients are briefly studied, and it is shown that the absolute value of this polynomial at -1 equals the odd-even invariant, in analogy with the usual chromatic polynomial and the number of acyclic orientations.

1. Introduction. The odd-even invariant of oriented matroids was introduced in [4]. The case in which the oriented matroid is graphic was considered in [3], and in the present paper it is given further consideration.

The formal definition appears in the next section. Making use of the well-known correspondence between oriented matroids and arrangements, the basic idea can be conveyed in geometrical terms as follows. An arrangement of (distinct) hyperplanes in $\mathbb{R}^d$ separates the space into open polyhedral cells, the connected components of the complement of the union of the hyperplanes. The "tope graph" (see [2]) of the arrangement (or the corresponding oriented matroid) has the cells as its vertices, with two cells adjacent provided that their closures have an intersection of codimension 1, so that they share a common border. This graph is bipartite and connected, and when the graph is 2-colored, red and blue, the odd-even invariant of the oriented matroid is (as in [4]) the absolute value of the difference between the number of red cells and the number of blue cells. In the graphic case

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(restricted in this description to graphs without loops or multiple edges), these full-dimensional cells of the graphic oriented matroid correspond to the acyclic orientations of the graph. Cells share a common border provided that the corresponding acyclic orientations differ by the reversal of a single edge. If an orientation ω is fixed, then the odd-even invariant of the graph G is the absolute value of the difference between the number of acyclic orientations agreeing with ω on an even number of edges and the number agreeing on an odd number of edges. For oriented matroids, it was proven in [4] that the odd-even invariant of an oriented matroid and of its dual are equal.

It is known since [5] that the number of acyclic orientations of a graph G equals $|P(-1)|$, where P is the chromatic polynomial of G. We introduce the "odd-even chromatic polynomial" P^{oe}. The absolute value of $P^{\text{oe}}(-1)$ equals the odd-even invariant. While the chromatic polynomial and the number of acyclic orientations of a graph are Tutte-Grothendieck invariants and depend only on the graphic matroid, this is not true of the odd-even chromatic polynomial and the odd-even invariant; and although the ordinary chromatic polynomial extends to arbitrary matroids in a fairly natural way, there is no similar extension of the odd-even chromatic polynomial to matroids or oriented matroids, as can be seen from the fact that different trees have isomorphic matroids but can have different odd-even chromatic polynomials.

The odd-even invariant in the graphic case was the subject of the first author's high school senior project. He presented his results in the school journal [3]. In particular, he found a generating function for the odd-even invariants of the complete bipartite graphs $K_{m,n}$. See Theorem 6 below. In sections 2 and 3 below, results from [3] are reiterated.

For background on oriented matroids, see [2]; however, no knowledge of oriented matroids will be required in what follows. Some familiarity with graph theory will be assumed. For this, [1] might be more than sufficient.

2. The odd-even invariant. We will be concerned with a fixed undirected graph and the directed graphs having it as their underlying undirected graph. The graphs considered are finite and they may have loops and parallel edges.

The vertex set of the (undirected) graph G is denoted by $V(G)$; its edge set is $E(G)$. An edge e of G that is not a loop may be given a direction; that is, one of its vertices may be designated as its *head*, with the other being its *tail*. There are two possible directions for e. The set of all directed edges is denoted by $D(G)$. The projection $\pi : D(G) \to E(G)$ is the function that "forgets" the direction. The digraphs having G as underlying graph correspond to the functions $\omega : E(G) \to D(G)$ such that $\pi(\omega(e)) = e$ for each $e \in E$ that is not a loop. We call such functions *orientations* of G and we denote the set of such functions ω by $\mathcal{B}$. An orientation $\omega \in \mathcal{B}$ is said to be *acyclic* if the corresponding digraph has no directed cycles. We put $\mathcal{A}(G) = \{\omega \in \mathcal{B} \mid \omega \text{ is acyclic }\}$. In most of what follows, a fixed element $\omega_0 \in \mathcal{B}$ will be specified, and then we will denote by $\mathcal{A}^+(G)$ the set of acyclic $\omega \in \mathcal{A}(G)$ such that $\omega(e) \neq \omega_0(e)$ for an even number of edges e, and by $\mathcal{A}^-(G)$ the complementary set $\mathcal{A}(G) \setminus \mathcal{A}^+(G)$, of those that disagree with ω_0 on an odd number of edges. The sets $\mathcal{A}^+(G)$ and $\mathcal{A}^-(G)$ depend upon the choice of ω_0, but only up to sign; that is, the partition $\{\mathcal{A}^+(G), \mathcal{A}^-(G)\}$ of $\mathcal{A}(G)$ does not vary with the choice of ω_0.

Given $\omega_0 \in \mathcal{B}$, the *odd-even invariant* $\text{œ}(G)$ is the absolute value of the difference, $|\mathcal{A}^+(G)| - |\mathcal{A}^-(G)|$. We call the difference itself the *signed* odd-even invariant: $\text{œ}(G, \omega_0) = |\mathcal{A}^+(G)| - |\mathcal{A}^-(G)|$; $\text{œ}(G) = |\text{œ}(G, \omega_0)|$. Reversing the direction of an edge of ω_0 changes the sign of the signed odd-even invariant; the odd-even invariant does not depend upon the choice of ω_0.

In the undirected graph G, edges are termed *parallel* if they are not loops and they are incident to the same two vertices. Being parallel is an equivalence relation on the set of edges that are not loops.

We will denote the graph obtained from G by deletion of a set $F \subseteq E(G)$ by $G \setminus F$, and that obtained by contraction by G/F.

The following theorem collects various simple facts about the odd-even invariant.

Theorem 1. *Suppose G is a graph, and $\omega_0 \in \mathcal{B}$.*

(a) If G has no edges then $\text{œ}(G) = \text{œ}(G, \omega_0) = 1$.

(b) If G has a loop then $\text{œ}(G, \omega_0) = 0$.

(c) If $|E(G)|$ is odd then $\text{œ}(G, \omega_0) = 0$.

(d) Suppose e and e' are parallel edges of G. Then, denoting by ω_0' the restriction of ω_0 to $E(G) \setminus \{e, e'\}$, $\text{œ}(G, \omega_0) = \text{œ}(G/\{e, e'\}, \omega_0') + \text{œ}(G \setminus \{e, e'\}, \omega_0')$.

(e) Suppose r, s, and t are distinct vertices of G, where s and t are not adjacent, and suppose that r is incident to precisely two edges, e and e', one of which is incident to s, and the other, to t. Then, denoting by

ω_0' the restriction of ω_0 to $E(G)\setminus\{e,e'\}$, $\text{œ}(G,\omega_0) = \text{œ}(G\setminus\{e,e'\},\omega_0') - \text{œ}(G/\{e,e'\},\omega_0')$.

(f) If G is the union of two subgraphs G' and G'' that share at most one vertex and ω_0', ω_0'' are the functions induced by ω_0 on $E(G')$, $E(G'')$ respectively, then $\text{œ}(G,\omega_0) = \text{œ}(G',\omega_0')\text{œ}(G'',\omega_0'')$.

Proof. (a): In this case, $|\mathcal{A}^+(G)| = 1$, $|\mathcal{A}^-(G)| = 0$.

(b): Here, $|\mathcal{A}^+(G)| = |\mathcal{A}^-(G)| = 0$.

(c): Reversing all edge directions is an involution on $\mathcal{A}(G)$. When $|E(G)|$ is odd, the contribution to $\text{œ}(G,\omega_0)$ of an element of $\mathcal{A}(G)$ is the negative of that of its reverse; so these contributions cancel.

(d): For any $\omega \in \mathcal{A}(G)$, e and e' have the same direction; that is, they have the same head and the same tail, by acyclicity. Any $\omega \in \mathcal{A}(G)$ can be obtained from an element of $\mathcal{A}(G \setminus \{e,e'\})$ by directing the edges e and e' one way or the other, and at least one of these two possibilities must yield an acyclic orientation. If both possibilities yield acyclic orientations, then the induced orientation of $G/\{e,e'\}$ is also acyclic.

(e): Let E_0 denote the set of edges of G not incident to r. Any orientation $\omega' : E_0 \to D_0$ extends to four orientations $\omega : E(G) \to D(G)$. These are not acyclic unless $\omega' \in \mathcal{A}(G \setminus \{e,e'\})$. If ω' is acyclic then two of the four extensions, namely those for which the edges incident to r both have r as head or both have r as tail, are acyclic. Additionally, both of the other orientations are also acyclic if $\omega' \in \mathcal{A}(G/\{e,e'\})$, and exactly one of them is acyclic if otherwise.

(f): Each acyclic orientation ω of G induces acyclic orientations ω' of G', ω'' of G'', and, since the two subgraphs share at most one vertex, the function taking ω to the pair (ω',ω'') is a one-to-one correspondence. We have

$$\begin{aligned}
\text{œ}(G,\omega_0) &= |\mathcal{A}^+(G)| - |\mathcal{A}^-(G)| \\
&= (|\mathcal{A}^+(G')||\mathcal{A}^+(G'')| + |\mathcal{A}^-(G')||\mathcal{A}^-(G'')|) \\
&\quad -(|\mathcal{A}^+(G')||\mathcal{A}^-(G'')| + |\mathcal{A}^-(G')||\mathcal{A}^+(G'')|) \\
&= (|\mathcal{A}^+(G')| - |\mathcal{A}^-(G')|)(|\mathcal{A}^+(G'')| - |\mathcal{A}^-(G'')|) \\
&= \text{œ}(G',\omega_0')\text{œ}(G'',\omega_0'').
\end{aligned}$$

□

Part (d) of Theorem 1 can be used to reduce the computation of the odd-even invariant to the case when no parallel class of edges has cardinality greater than 2. Also it, combined with part (b), shows that if G has more than two edges in some parallel class, then its odd-even invariant is the same as that of the graph obtained by deleting two of those edges. Parts (e) and (f) enable the computation of the odd-even invariant for any graph that can be built up from a graph with known odd-even invariant by successively adding vertices of degree at most 2. Using part (f) and the fact that the graph having two vertices and one edge connecting them has invariant 0, it follows that any graph with a leaf has odd-even invariant 0, as does any graph with a bridge.

The *neighborhood* of the vertex v of G is the subgraph induced by v together with all vertices of G that are adjacent to v. A *clique* in G is a set of vertices of G having the property that each pair of distinct vertices of the set comprise the vertices of at least one edge of G, and no vertex of the set is on a loop of G. A vertex is *simple* if it is not on a loop and is incident to no parallel edges. Also we denote the number of edges incident to v by $\deg(v)$.

THEOREM 2. *Suppose v is a simple vertex of G which is not the tail of any edge, with respect to the orientation ω_0, and suppose the neighborhood of v is a clique. Let E_0 denote the set of edges incident to v and let ω_0' denote the restriction of ω_0 to $E \setminus E_0$. If $\deg(v)$ is even then $œ(G, \omega_0) = œ(G \setminus E_0, \omega_0')$; if $\deg(v)$ is odd then $œ(G, \omega_0) = 0$.*

Proof. Let U denote the set of vertices to which v is adjacent. By assumption, U induces a clique of G, so, any $\omega' \in \mathcal{A}(G \setminus E_0)$, induces a linear ordering on U. Therefore, for any such ω', we may write $U = \{u_1, u_2, \ldots, u_d\}$, where $d = \deg(v)$, with the indices respecting the ordering induced on U, so that, for $1 \leq i < j \leq d$, each edge incident to u_i and u_j has u_i as tail and u_j as head. Each $\omega \in \mathcal{A}(G)$ induces an orientation $\omega' \in \mathcal{A}(G \setminus E_0)$, and each $\omega' \in \mathcal{A}(G \setminus E_0)$ extends to exactly $d + 1$ orientations $\omega \in \mathcal{A}(G)$: For each k with $0 \leq k \leq d$, there is an acyclic orientation of G extending ω' for which the edges joining $u_1, \ldots, u_k$ to v each have v as head, while those joining $u_{k+1}, \ldots, u_d$ to v have v as tail, where the indexing on the u_i's is determined by ω' as above. This yields a sequence of $d + 1$ orientations $\omega \in \mathcal{A}(G)$, each of which agrees with ω' on $E \setminus E_0$, and they alternately agree with an odd or with an even number of edges of E_0, so that the net contribution to $\text{œ}(G, \omega_0)$ is 0 if d is odd. When d is even, and if the number of edges on which ω' disagrees with ω_0 is odd, then the number of edges disagreeing altogether alternates, odd, even, odd, ..., odd, so that the net contribution of the $d+1$ orientations to $\text{œ}(G, \omega_0)$ is -1. If the number of edges on which ω' disagrees with ω_0 is even, then the number of edges disagreeing alternates, even, odd, even, ..., even, so that the contribution of the $d+1$ orientations to $\text{œ}(G, \omega_0)$ is 1. Therefore the contribution is the same as that of ω' to $\text{œ}(G \setminus E_0, \omega'_0)$. $\square$

The next result gives a recursive method for computing $\text{œ}(G)$. Given a vertex x of G, let $\mathcal{A}_x$ denote the set of acyclic orientations having x as a terminal vertex. For $I \subseteq V$, let $\mathcal{A}_I$ denote the set of acyclic orientations in which all vertices of I are terminal; $\mathcal{A}_I = \emptyset$ unless I is an independent set. Let $\nu(I, \omega_0)$ denote the number of edges e having an element of I as its initial vertex in ω_0; it is the number of edges leaving I.

Theorem 3. *The signed odd-even invariant is given by the following summation over the independent sets of G.*

$$\text{œ}(G, \omega_0) = \sum_{\substack{I \subseteq V(G), \\ I \text{ independent}}} (-1)^{|I|-1+\nu(I,\omega_0)} \text{œ}(G \setminus I, \omega_0|_{E(G \setminus I)}).$$

Proof. This is immediate from the principle of inclusion-exclusion, $\mathcal{A}(G)$ being $\bigcup_{x \in V(G)} \mathcal{A}_x$ and, for $I \subseteq V(G)$, $\mathcal{A}_I$ being $\bigcap_{x \in I} \mathcal{A}_x$. $\square$

Theorem 4. *If G is a chordal graph having at least one edge, then $\text{œ}(G) = 0$. If G is a circuit with n vertices or a wheel with $n+1$ vertices (and, therefore, n spokes) then*

$$\text{œ}(G) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Proof. These statements follow as easy corollaries of the previous theorems. A chordal graph G can be built from scratch by successively adding one new vertex at a time, with the new vertex being adjacent to the vertices of a clique in the previous graph; and if G has an edge, we may start building from that edge. That $\text{œ}(G) = 0$ then follows inductively using Theorem

2. If G is a circuit with n vertices then $\text{œ}(G) = 0$ if n is odd, by (c) of Theorem 1; and if n is even then the fact that $\text{œ}(G) = 2$ follows inductively, beginning the induction at the 2-vertex circuit, using part (e) of Theorem 1, and noting that deleting a vertex yields a chordal graph. In the case of a wheel graph with hub v and $n + 1$ vertices, the only nonzero term in the summation of Theorem 3 is that in which $I = \{v\}$, since deletion of any other independent set yields a chordal graph. When v is deleted, what remains is a circuit with n vertices. □

3. The odd-even invariants of the complete bipartite graphs. For integers $m, n \geq 0$, let $K_{m,n}$ be the complete bipartite graph having vertex set $V = U_1 \cup U_2$, where $|U_1| = m$, $|U_2| = n$, and having mn edges, joining the m vertices of U_1 and the n vertices of U_2. Let ω_0 denote the orientation in which the vertices of U_2 are terminal, and put $b_{m,n} = \text{œ}(K_{m,n}, \omega_0)$. Based upon Theorem 3 we derive the following recurrence relation for the numbers $b_{m,n}$.

THEOREM 5. *The $b_{m,n}$'s satisfy the following recurrence relation:*

$$b_{m,n} = \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} b_{m,n-i} + \sum_{j=1}^{m} (-1)^{nj+j-1} \binom{m}{j} b_{m-j,n}.$$

The equation holds for $m, n \geq 0$, not both 0.

Proof. This follows directly from Theorem 3 upon noting that the nonempty independent sets in $K_{m,n}$ are the nonempty subsets of U_1 together with the nonempty subsets of U_2. The first summation is the contribution from the nonempty subsets of U_2. Since all vertices of such a set are terminal in ω_0, the value of ν of Theorem 3 is 0. The second summation is the contribution from the nonempty subsets I of U_1, where j represents $|I|$. Since all the vertices are initial, each edge connecting the set to U_2 is reversed; ν is the number of such edges, which is nj. □

The following table, containing the values of $b_{m,n}$ for small m, n, was obtained using Theorem 5 with the boundary values $b_{m,0} = b_{0,n} = 1$.

$m,n:$	0	1	2	3	4	5	6
0 :	1	1	1	1	1	1	1
1 :	1	0	0	0	0	0	0
2 :	1	0	-2	-6	-14	-30	-62
3 :	1	0	-6	0	-6	0	-6
4 :	1	0	-14	-6	118	690	2806
5 :	1	0	-30	0	690	0	3570
6 :	1	0	-62	-6	2806	3570	-39722

The next theorem presents the exponential generating function for these numbers. For background on generating functions in combinatorics consult, for example, [7].

THEOREM 6. *The exponential generating function for the $b_{m,n}$'s is:*

$$f(x,y) = \sum_{m,n\geq 0} \frac{b_{m,n}}{m!n!} x^m y^n = \frac{e^x + e^y - 1}{2(1-\cosh x)(1-\cosh y)+1}.$$

Proof. Given the formal power series

$$A = \sum_{m,n} \frac{\alpha_{m,n}}{m!n!} x^m y^n \text{ and } B = \sum_{m,n} \frac{\beta_{m,n}}{m!n!} x^m y^n,$$

the product $C = AB$ is

$$C = \sum_{m,n} \frac{\gamma_{m,n}}{m!n!} x^m y^n,$$

where

$$\gamma_{m,n} = \sum \binom{m}{i}\binom{n}{j} \alpha_{i,m-i}\beta_{j,n-j}.$$

Using this, we find that, for $m,n \geq 0$, not both 0, the coefficient of $x^m y^n$ in $f(x,y)(1-e^{-y})$ is $\frac{1}{m!n!}\sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}b_{m,n-i}$; that in $f(x,y)(1-\cosh x)$ is $\frac{1}{m!n!}\sum_{j=1}^{m}\frac{1}{2}((-1)^{j-1}-1)\binom{m}{j}b_{m-j,n}$; and that in $f(x,-y)\sinh x$ is $\frac{1}{m!n!}\sum_{j=1}^{m}\frac{1}{2}((-1)^{n}+(-1)^{n+j-1})\binom{m}{j}b_{m-j,n}$.

Summing, and using Theorem 5 and the fact that $\frac{1}{2}\big((-1)^{n+j-1} - 1 + (-1)^{j-1} - (-1)^{n-1}\big) = (-1)^{nj+j-1}$, we find that $f(x,y) = 1 + f(x,y)(1 - e^{-y}) + f(x,y)(1 - \cosh x) + f(x,-y)\sinh x$. Upon rewriting, this is

$$-f(x,-y)\sinh x = 1 - f(x,y)(e^{-y} + \cosh x - 1).$$

Also, equivalently upon replacing y by $-y$ in the above,

$$f(x,-y)(e^{y} + \cosh x - 1) = 1 + f(x,y)\sinh x.$$

Combining the two preceding equations we have

$$\begin{aligned}
&(-1 + f(x,y))(e^{-y} + \cosh x - 1)(e^{y} + \cosh x - 1)\\
&\quad = f(x,-y)\sinh x(e^{y} + \cosh x - 1)\\
&\quad = \sinh x(1 + f(x,y)\sinh x).
\end{aligned}$$

Then

$$\begin{aligned}
f(x,y)\big((e^{-y} + \cosh x - 1)(e^{y}+\cosh x - 1) - \sinh^2 x\big)&\\
= \sinh x + e^{y} + \cosh x - 1,&
\end{aligned}$$

which upon simplification becomes

$$f(x,y)\big(2(1 - \cosh x)(1 - \cosh y) + 1\big) = e^{x} + e^{y} - 1.$$

□

4. The odd-even chromatic polynomial. The *odd-even chromatic polynomial* P^{oe} bears a relation to the odd-even invariant that parallels that of the (ordinary) chromatic polynomial to the number of acyclic orientations of a graph. Consider two functions, P^e_G and P^o_G, defined as follows. Given an orientation ω_0 of G, let P^e_G (P^o_G) be the function on positive integers whose value at k is the number of functions $f : V(G) \to [k]$ such that (1) no two adjacent vertices have the same value (that is, it is a $[k]$-*coloring*), and (2) the number of edges e such that $f(v) < f(u)$, where u is the tail and v is the head of e, is even (respectively, odd). Then, when k is a positive integer, the total number of k-colorings of G is a function $P_G(k) = P^e_G(k) + P^o_G(k)$. $P_G(k)$ is a polynomial in k, the usual *chromatic* *polynomial* of G. As we will see, $P^e_G(k)$ and $P^o_G(k)$ are also polynomials in k. We define $P^{oe}_G(k) = P^e_G(k) - P^o_G(k)$. This depends up to sign on ω_0, but we will usually suppress this dependence in the notation. Given a coloring f, we define the *sense* of the coloring to be *even* if the number of edges e having tail u and head v, where $f(u) > f(v)$, is even, and *odd*, otherwise. Given k, the odd-even chromatic polynomial gives the number of k-colorings having even sense minus that of those having odd sense.

As an example, consider $G = K_{1,2}$, with ω_0 being the orientation in which both edges have the vertex of degree 1 as their tail. In this case it is not hard to verify the following:

$$P^e_G(k) = \frac{2}{3}k^3 - k^2 + \frac{1}{3}k,$$

$$P^o_G(k) = \frac{1}{3}k^3 - k^2 + \frac{2}{3}k,$$

$$P^{oe}_G(k) = \frac{1}{3}k^3 - \frac{1}{3}k, \quad \text{and}$$

$$P_G(k) = k^3 - 2k^2 + k.$$

Replacing the orientation ω_0 by one having the direction of one edge changed results in the reversal of P^e_G and P^o_G, sign change for P^{oe}_G, and no change for P_G.

If a graph G has an odd number of edges then $P^{oe}_G(k) = 0$. If G is the disjoint union of two subgraphs G_1, G_2, then P^{oe}_G the product of the odd-even chromatic polynomials of the two subgraphs. If $n > 1$ and $G = K_n$ then $P^{oe}_G(k) = 0$.

As is well-known, $P_G(k)$ is a polynomial in k. The following theorem gives the analogous fact for P^{oe}_G. The proofs given for the next two theorems make use of facts about Ehrhart polynomials of convex polytopes. See Stanley [7], pages 235–241, for basic facts about Ehrhart polynomials used here. These statements can also be proven easily using Theorem 10, below, without reference to Ehrhart polynomials.

THEOREM 7. *The functions P^e_G and P^o_G are polynomials of degree $|V(G)|$; and P^{oe}_G is a polynomial of degree at most $|V(G)|$.*

Proof. Each coloring f of G yields an orientation ω of G by taking, for edge e incident to vertices u and v with $f(u) < f(v)$, $\omega(e)$ to be the orientation in which u is the tail and v is the head. The resulting orientation is acyclic, so this yields a partition of the collection of colorings into classes corresponding to the acyclic orientations of G. Given an acyclic orientation ω, let $Q_\omega(k)$ denote the number of $[k]$-colorings of G that yield ω in this way.

The functions Q_ω are related to the Ehrhart polynomials of certain convex polytopes $\mathcal{Q}$. Suppose that an acyclic orientation ω is given. It determines a partial ordering of the vertex set of G. The set $\mathcal{Q}$ of functions $f : V(G) \to [0, 1]$ such that, if the edge $e \in E(G)$ has tail u and head v then $f(u) \leq f(v)$, is (following Stanley [6]) the *order polytope* of the poset. The polytope $\mathcal{Q}$ has dimension $d = |V(G)|$. Consider the dilation $k\mathcal{Q}$ of $\mathcal{Q}$ by a factor of k, where k is a positive integer. The Ehrhart polynomial of a polytope $\mathcal{Q}$ having integer vertices is the function $\mathcal{E}$ whose value $\mathcal{E}(k)$ is the number of points of $k\mathcal{Q}$ that have integer coordinates (for $k = 1, 2, \dots$). This function is a polynomial in k whose degree is the dimension of $\mathcal{Q}$. The value of this polynomial at 0 is 1. Its values at negative integers also have combinatorial significance: the number of points having integer coordinates that lie in the interior of $k\mathcal{Q}$ (for $k > 0$) is given by $(-1)^d\mathcal{E}(-k)$, where d is the dimension of the polytope, according to the Combinatorial Reciprocity Theorem. Since $Q_\omega(k)$ is the number of points having integer coordinates that lie in the interior of the convex polytope $(k+1)\mathcal{Q}$, it follows that $Q_\omega(k)$ is a polynomial of degree $|V(G)|$.

The validity of the theorem follows by noting that

$$P^e_G(k) = \sum_{\substack{\omega \in \mathcal{A}(G),\\ d(\omega,\omega_0) \text{ even}}} Q_\omega(k),$$

$$P^o_G(k) = \sum_{\substack{\omega \in \mathcal{A}(G),\\ d(\omega,\omega_0) \text{ odd}}} Q_\omega(k),$$

and $P^{\text{œ}}_G(k) = P^e_G(k) - P^o_G(k)$. ◻

THEOREM 8. *We have* $\text{œ}(G, \omega_0) = (-1)^{|V(G)|} P^{\text{œ}}_G(-1)$, *where* ω_0 *is the orientation used to obtain* $P^{\text{œ}}_G$.

Proof. In the proof of Theorem 7, the Ehrhart polynomials $\mathcal{E}$ have value $\mathcal{E}(0) = 1$. When $k = -1$ the values $Q_\omega(k)$ appearing in the summations are $Q_\omega(-1) = (-1)^{|V(G)|}\mathcal{E}(0) = (-1)^{|V(G)|}$. ◻

The following theorem is an analogue of the well-known reduction theorem for the chromatic polynomial. It implies Theorem 1, part (d), by the preceding theorem; and the proof, which is similar, is omitted.

THEOREM 9. *Suppose e and e' are parallel edges of G. If, in ω_0, the edges e, e' have the same head (and the same tail) then $P^{\text{oe}}_G = P^{\text{oe}}_{G\setminus\{e,e'\}} + P^{\text{oe}}_{G/\{e,e'\}}$.*

Theorem 9 can be used to reduce the computation of the odd-even chromatic polynomial to that of graphs without multiple edges. Also, writing the equation as $P^{\text{oe}}_{G\setminus\{e,e'\}} = P^{\text{oe}}_G - P^{\text{oe}}_{G/\{e,e'\}}$, the computation of the polynomial can be recursively reduced to the computations involving only graphs G for which $V(G)$ is a clique. It is easy to see that for such a clique-graph the odd-even chromatic polynomial is simply $\gamma\binom{k}{n}$, where $n = |V(G)|$ and γ is the number of $[n]$-colorings of even sense minus the number of $[n]$-colorings of odd sense. We will consider this further in the next section.

The following theorem is an analogue of Theorem 3 for the odd-even chromatic polynomials.

THEOREM 10. *The values of the odd-even chromatic polynomial are given recursively by the following sum over the independent sets of G:*

$$P^{\text{oe}}_G(k) = \sum_{I\in\mathcal{I}(G)} (-1)^{\nu(I)} P^{\text{oe}}_{G\setminus I}(k-1).$$

Proof. Given a $[k]$-coloring of G, let I be the set of vertices of color k. This is an independent set. Clearly the contribution to $P^{\text{oe}}_G(K)$ of $[k]$-colorings of G in which precisely the vertices of I have color k is $(-1)^{\nu(I)}P_{G\setminus I}(k-1)$, with the factor $(-1)^{\nu(I)}$ taking care of the sign. $\square$

For $G = K_{m,n}$, and letting U_1, U_2 be as before, with ω_0 being the orientation in which all edges are directed from U_1 to U_2, denote the odd-even chromatic polynomial P^{oe}_G by $B_{m,n}$.

THEOREM 11. *When not both m and n are odd, the following equation holds.*

$$\begin{aligned} &B_{m,n}(k+1) - B_{m,n}(k) \\ &\quad = \sum_{j=1}^{n}\binom{n}{j}B_{m,n-j}(k) + \sum_{i=1}^{m}\binom{m}{i}B_{m-i,n}(k). \end{aligned}$$

Proof. Directly from Theorem 10 we get:

$$\begin{aligned} &B_{m,n}(k+1) - B_{m,n}(k) \\ &\quad = \sum_{j=1}^{n} \binom{n}{j} B_{m,n-j}(k) + \sum_{i=1}^{m} \binom{m}{i} (-1)^{ni} B_{m-i,n}(k). \end{aligned}$$

When n is even, $(-1)^{ni} = 1$, and we obtain the equation of the theorem. When n is odd and m is even, $(-1)^{ni} = -1$ when i is odd, in which case $m - i$ is odd, so that $B(m-i, n)$ is 0. $\square$

Theorem 11 can be used to construct a table of the polynomials $B_{m,n}$, expressed in terms of the polynomials $\binom{k}{d}$ $(d = 0, 1, \dots)$. Here is a small portion of this table.

m, n :	0	1	2
0 :	1	$\binom{k}{1}$	$2\binom{k}{2} + \binom{k}{1}$
1 :	$\binom{k}{1}$	0	$2\binom{k}{3} + 2\binom{k}{2}$
2 :	$2\binom{k}{2} + \binom{k}{1}$	$2\binom{k}{3} + 2\binom{k}{2}$	$8\binom{k}{4} + 12\binom{k}{3} + 2\binom{k}{2}$
3 :	$6\binom{k}{3} + 6\binom{k}{2} + \binom{k}{1}$	0	$24\binom{k}{5} + 48\binom{k}{4} + 20\binom{k}{3} + 2\binom{k}{2}$

To obtain the table, first put a 1 as the entry labeled by $m = 0$ and $n = 0$. Put 0's in the positions indexed by odd m and n. Then, for each of the other entries (starting in the corner and working outward), compute the sum along the row and column containing that entry as in the right side of the equation of the theorem, obtaining a sum of terms $\gamma\binom{k}{d}$. Finally, advance the d's, so that the term becomes $\gamma\binom{k}{d+1}$, thereby obtaining the proper entry for that position. Notice that the coefficients are nonnegative (using this basis, and for our choice of ω_0). It isn't hard to verify that, in the entry for $K_{m,n}$, the coefficient of $\binom{k}{m+n}$ is $m!n!\binom{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor}$, when m and n are not both odd.

5. The coefficient of $\binom{k}{l}$. Let $\gamma_l(G, \omega_0)$ denote the coefficient of $\binom{k}{l}$ in $P_G^{\infty}(k)$. Putting $n = |V(G)|$, we may write

$$P_G^{\infty}(k) = \sum_{l=0}^{n} \gamma_l(G, \omega_0) \binom{k}{l}.$$

We briefly consider the coefficients $\gamma_l(G, \omega_0)$.

The coefficient $\gamma_0(G, \omega_0)$ of $\binom{k}{0}$ equals 0 (unless G has no vertices). For $1 \leq l \leq n$, let Π_l denote the set of ordered partitions $(I_1, \ldots, I_l)$ of $V(G)$ into l nonempty independent sets. Each k-coloring $f : V(G) \to [k]$ of G determines an element of Π_l, where l is the cardinality of the image $f(V(G))$: If $k_1 < k_2 < \cdots < k_l$ are the "colors" that are actually used, then the element of Π_l is $(f^{-1}(k_1), f^{-1}(k_2), \ldots, f^{-1}(k_l))$. Given $\pi = (I_1, \ldots, I_l) \in \Pi_l$, let G_π denote the graph having vertices $I_1, \ldots, I_l$, with edges arising from edges of G: An edge $e \in E(G)$ incident to vertices $u, v \in V(G)$, where $u \in I_a$ and $v \in I_b$, is also considered to be an edge of G_π, incident to the vertices I_a and I_b of G_π. If ω_0 is an orientation of G, it is also an orientation of G_π. The graphs G_π (for $\pi \in \Pi_l$) have l vertices.

Theorem 12. *The coefficient γ_l in $P_G^{\text{œ}}(k) = \sum_l \gamma_l \binom{k}{l}$ is*

$$\gamma_l(G, \omega_0) = \sum_{\pi \in \Pi_l} \gamma_l(G_\pi, \omega_0).$$

Proof. $P_G^{\text{œ}}(k)$ is the sum over all k-colorings of the sense, ± 1, of the coloring. Given a partition $\pi \in \Pi_l$, the set $\{k_1, \ldots, k_l\}$ can be chosen in $\binom{k}{l}$ distinct ways, and for each such choice, the contribution to $P_G^{\text{œ}}(k)$ of the k-colorings for which π is the associated partition is $\gamma_l(G_\pi, \omega_0)$. □

This result motivates an effort to determine the coefficient of the term of degree $|V(G)|$; we are after $\gamma_n(G, \omega_0)$, where $n = |V(G)|$. This is simply the sum over the permutations π of the vertex set of G of numbers $\alpha_\pi = \pm 1$, where $\alpha_\pi = 1$ if the orientation induced by the permutation agrees with ω_0 on an even number of edges, and $\alpha_\pi = -1$, otherwise.

Put $V(G) = \{v_1, \ldots, v_n\}$. We assume henceforth that ω_0 is induced by the order given by the indexing of the vertices, so that for an edge e incident to vertices v_i and v_j where $i < j$, $\omega_0(e)$ has tail v_i and head v_j.

If the graph G has a loop then $\gamma_l(G, \omega_0) = 0$; we assume G has no loops.

Let R be the following ring. For each vertex v_i, let x_i denote an indeterminate. As an abelian group the ring is freely generated by the square-free monomials in the x_i's. There are the following multiplicative identities: $x_i^2 = 0$, for each i; and $x_i x_j = -x_j x_i$ if an odd number of edges of G oppose the direction v_i to v_j, where $i < j$; otherwise, $x_i x_j = x_j x_i$.

Call a permutation $\pi = (v_1, \ldots, v_n)$ of the vertex set an *even ordering* if, for each index i, the number of $j < i$ such that v_i and v_j are adjacent is even.

THEOREM 13. *The highest-degree coefficient is $\gamma_n(G, \omega_0) = \gamma$, where $(x_1 + \ldots + x_n)^n = \gamma x_1 \cdots x_n$. Furthermore, $\gamma = \sum_\pi N(\pi)$, where the sum is taken over all permutations of the vertex set of G, $N(\pi) = 0$ if π is not an even ordering, and otherwise $N(\pi) = (-1)^s$, where $s = s(\pi)$ is the number of edges e that are incident to vertices v_i and v_j with $i < j$ and $\pi(i) > \pi(j)$.*

Proof. The expression $(x_1 + \ldots + x_n)^n$ is the sum of the $n!$ terms of the form $x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(n)}$, one for each permutation π of $\{1, 2, \ldots, n\}$. In R, $x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(n)} = \alpha x_1 x_2 \cdots x_n$, where $\alpha = 1$ if there is an even number of edges e incident to vertices v_i and v_j for which $i < j$ and $\pi(i) > \pi(j)$, and it is -1 otherwise. Summing yields the first statement.

We show that

$$(x_1 + \ldots + x_n)^n = (\sum_\pi N(\pi)) x_1 x_2 \cdots x_n.$$

This clearly holds if $n = 1$. To see that this is the case in general, assume inductively that n is an integer that is greater than 1 and equality holds for subgraphs having $n - 1$ vertices. Consider

$$(x_1 + \ldots + x_n)^n = (x_1 + \ldots + x_n)^{n-1}(x_1 + \ldots + x_n).$$

Notice that, for any i,

$$(x_1 + \ldots + x_n)^{n-1} x_i = (x_1 + \ldots + x_{i-1} + x_{i+1} + \ldots + x_n)^{n-1} x_i,$$

since $x_i^2 = 0$. It is clear that the contribution of this last product to the coefficient is the odd-even invariant of the subgraph induced by $\{v_j \mid j \neq i\}$=(with the orientation induced by ω_0) multiplied by $(-1)^t$, where t is the number of vertices adjacent to v_i and appearing after it in π. This subgraph has an odd number of edges if v_i has odd degree, so the contribution in that case is 0. Summing over the vertices gives the inductive step. $\square$

THEOREM 14. *For a bipartite graph G, $|\gamma_n(G, \omega_0)|$ is the number of linear orderings of the vertex set in which each vertex is adjacent to an even number of its predecessors.*

Proof. For any two even orderings, π_1 and π_2, and for any vertex v_i, the number $\delta(v_i)$ of edges incident to v_i for which the orientations induced by π_1 and π_2 differ is even. To see that this is true, let a denote the number of edges incident to v_i that are directed toward v_i by both orderings, let b be the number directed away from v_i by π_1 and toward v_i by π_2, and let c be the number directed away from v_i π_2 and toward v_i by π_1; then $a+b$ and $a+c$ are both even, so $b+c$, which is the number of edges on which the orientations differ, is also even.

Since G is bipartite, there exists a 2-coloring of its vertices. Given the 2-coloring, the total number of edges on which π_1 and π_2 induce opposite orientations can be found by summing the $\delta(v_i)$'s over the vertices of one color, so the total number of edges for which this is the case must also be even; that is, the numbers $s(\pi_1)$ and $s(\pi_2)$, are congruent modulo 2. $\square$

The k-th Eulerian number of order n, $E(n,k)$, is the number of permutations having k "falls," a *fall* in a permutation $\pi : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ being a pair of consecutive integers $i, i+1$, with $1 \le i \le n-1$, such that $\pi(i+1) < \pi(i)$.

THEOREM 15. *For a path G with n vertices, $|\gamma_n(G,\omega_0)|$ is the absolute value of the alternating sum of the Eulerian numbers of order n.*

Proof. We may take G to be the path with vertex set $\{1,2,\ldots,n\}$ and edges $\{1,2\},\{2,3\},\ldots,\{n-1,n\}$. For ω_0, we take the orientation for which each edge $\{i,i+1\}$ is directed toward $i+1$. The alternating sum of the Eulerian numbers is $E(n,0)-E(n,1)+\ldots+(-1)^{n-1}E(n,n-1)$. Clearly this is the number of orderings of $\{1,2,\ldots,n\}$ with the order agreeing with ω_0 on an even number of edges of the path minus the number agreeing on an odd number of edges. $\square$

Of course in Theorem 15, when n is even, the value is 0.

Some notes and acknowledgments. There are many questions that one might ask about the graphical invariants introduced here. We mention a few of these in the next paragraphs.

The algorithmic computational complexity of computing these invariants is certainly of interest, although it has not been discussed here.

It is apparent that the odd-even invariant often has the value 0. Is there anything "special" about the graphs for which this is true? Perhaps some simple property of the graph might make it easy to determine whether or not this is the case.

The $B_{m,n}$'s of Theorem 11 are linear combinations of the $\binom{k}{l}$'s, having nonnegative coefficients. For what other graphs does this happen?

We know that if graphs G and $\widetilde{G}$ are planar duals, then their odd-even invariants are equal. Is there some simple relationship between their odd-even chromatic polynomials?

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