Some Review

Ω - the sample space

$P(E)$

$X : \Omega \to \mathbb{R}$ - a random variable

$P(a \le X \le b)$ means

$P(\{\omega \in \Omega : a \le X(\omega) \le b\})$.

X - a discrete random variable

Definition. Expected Value.

$$E(X) = \sum_x P(X=x)\, x .$$

Theorems. $E(X+Y) = E(X) + E(Y)$.

$$E(cX) = c\,E(X).$$

[That's linearity. But $E(XY)$ needn't equal $E(X)E(Y)$.]

$$E(g(X)) = \sum_x P(X=x)\, g(x) .$$

Definition. Variance.

$$\operatorname{var}(X) = E\left((X-\mu)^2\right),$$

where $\mu = E(X)$.

Theorem. $\operatorname{var}(X) = E(X^2) - E(X)^2$.

Proof. $\operatorname{var}(X) = E\left((X - E(X))^2\right)$

$$= E\left(X^2 - 2E(X)X + E(X)^2\right)$$

$$= E(X^2) - E(2E(X)X) + E(X)^2$$

$$= E(X^2) - 2E(X)\cdot E(X) + E(X)^2$$

$$= E(X^2) - E(X)^2. \qquad \square$$

Definition. Standard deviation.

$$\sigma(X) = \sqrt{\operatorname{var}(X)}.$$

<u>Example</u>. A random variable W has three possible values, namely, 0, 1, and 4, equal likely.

What are the expected value and variance of W ?

<u>Another</u> <u>example</u>. Y has the possible values -1, 0, and 1, with $P(-1) = P(1) = \frac{1}{4}$ and $P(0) = \frac{1}{2}$. What are the expected value and variance of Y?

Solutions.

w	0	1	4
$P(W=w)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$$E(W) = 0\cdot\frac{1}{3} + 1\cdot\frac{1}{3} + 4\cdot\frac{1}{3} = \frac{5}{3}.$$

$$E(W^2) = 0\cdot\frac{1}{3} + 1\cdot\frac{1}{3} + 16\cdot\frac{1}{3} = \frac{17}{3}.$$

$$\text{var}(W) = \frac{17}{3} - \frac{25}{9} = \frac{51}{9} - \frac{25}{9} = \frac{26}{9}.$$

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y	-1	0	1
$P(Y=y)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$$E(Y) = -1\cdot\frac{1}{4} + 0\cdot\frac{1}{2} + 1\cdot\frac{1}{4} = 0.$$

$$E(Y^2) = 1\cdot\frac{1}{4} + 0\cdot\frac{1}{2} + 1\cdot\frac{1}{4} = \frac{1}{2}.$$

$$\text{var}(Y) = \frac{1}{2}.$$

Definitions. Probability mass function of X.

$$x \mapsto P(X = x)$$

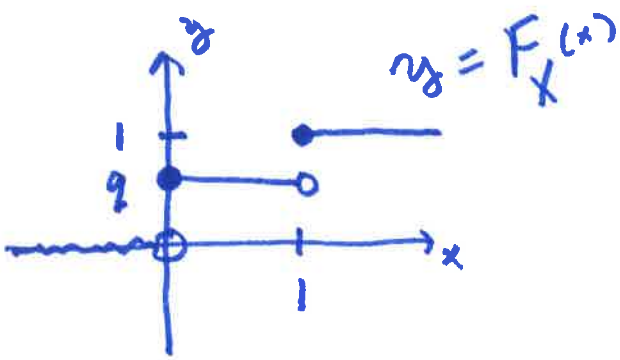
Distribution function of X.

$$F_X(x) = P(X \le x).$$

Example. Coin flipping. $X = 0$ or 1.

x	0	1
$P(X=x)$	$1-p$ (q)	p

← mass function.



← distribution function

Getting F_X from the mass function:

$$F_X(x) = P(X \le x) = \sum_{u:\, u \le x} P(X = u).$$

Getting probability mass function from distribution function:

$$P(X = x) = F_X(x) - \lim_{u \to x^-} F_X(u).$$

Properties of distribution function:

1. Increasing.
2. $\lim_{x \to -\infty} F_X(x) = 0$,
3. $\lim_{x \to \infty} F_X(x) = 1$.

Independence of random variables

Independence of events A, B:

$$P(AB) = P(A)P(B).$$

If X and Y are random variables (on the same probability space), then

Definition. Independence of r.v.'s:

X and Y are independent if, whenever A is an event determined by X (such as in $a \le X \le b$) and B is an event determined by Y (like $c \le Y \le d$) then A and B are independent.

[So, $P(a \le X \le b \text{ and } c \le Y \le d)$

$$= P(a \le X \le b)\, P(c \le Y \le d).$$]

For discrete r.v.'s X and Y

this implies

$$P(X=x \text{ and } Y=y) = P(X=x)\,P(Y=y),$$

which is equivalent.

Theorem. The following statements about the discrete r.v.'s X and Y are equivalent.

(a) X and Y are independent.

(b) For any real numbers a and b, $P(X=a \text{ and } Y=b)$
$= P(X=a)\cdot P(Y=b)$.

(c) For all such a, b, $P(X \le a \text{ and } Y \le b)$
$= F_X(a)\,F_Y(b)$.

(d) For any functions, g and h, the random variables $g(X)$ and $h(Y)$ are independent.

Independence of r.v.'s $X_1, \ldots, X_n$ is defined similarly. If $A_1, \ldots, A_n$ are events defined by $X_1, \ldots, X_n$ respectively then $A_1, \ldots, A_n$ are independent.

If so then, for real numbers $a_1, \ldots, a_n$,

$$P(X_1 \le a_1 \text{ and} \ldots \text{and } X_n \le a_n) = \prod_{i=1}^{n} P(X_i \le a_i).$$

Examples. Flipping two coins, rolling a die and flipping two coins, etc.

Theorem. If X and Y are independent random variables then $E(XY) = E(X)E(Y)$.

Corollary. If the r.v.'s X and Y are independent then $\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y)$, and

$$\sigma(X+Y) = \sqrt{\sigma^2(X) + \sigma^2(Y)}.$$

Proof of corollary:

$$\mathrm{var}(X+Y) = E((X+Y)^2) - E(X+Y)^2$$

$$= E(X^2+2XY+Y^2) - (E(X)+E(Y))^2$$

$$= E(X^2) + 2E(XY) + E(Y^2) - E(X)^2 - 2E(X)E(Y) - E(Y)^2$$

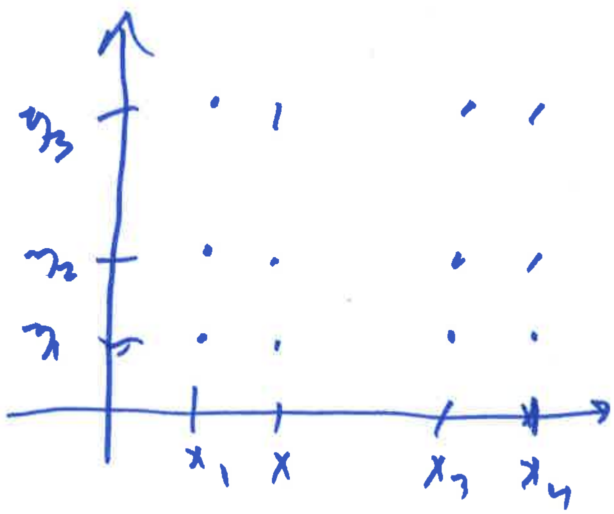
by independence

$$= E(X^2) + 2E(X)E(Y) + E(Y^2) - E(X)^2 - 2E(X)E(Y) - E(Y)^2$$

$$= E(X^2) - E(X)^2 + E(Y^2) - E(Y)^2$$

$$= \mathrm{var}(X) + \mathrm{var}(Y). \quad \square$$

THEOREM. If X_1 and X_2 are independent random variables then $E(X_1X_2) = E(X_1)E(X_2)$.



(Discrete case.)

$$E(XY) = \sum_i \sum_j P(X = x_i, Y = y_j)\, x_i y_j$$

$$= \sum_i \sum_j P(X = x_i) P(Y = y_j)\, x_i y_j$$

$$= \left(\sum_i P(X = x_i)\, x_i\right)\left(\sum_j P(Y = y_j)\, y_j\right)$$

$$= E(X)\,E(Y).$$

Some Famous Discrete Probability Distributions

Bernoulli

Binomial

Discrete uniform

Poisson

Geometric

Negative binomial

Hypergeometric

Binomial Distributions

parameters p, n

$0 \le p \le 1$, $n \in \{0, 1, 2, \ldots\}$

x	$P(X=x)$
0	$(1-p)^n$
1	$n\,p\,(1-p)^{n-1}$
2	$\binom{n}{2} p^2 (1-p)^{n-2}$
$\vdots$	$\vdots$
k	$\binom{n}{k} p^k (1-p)^{n-k}$
$\vdots$	$\vdots$
n	p^n

Uniform Discrete Distribution

on $[a,b]$ $(a, b \in \mathbb{Z}, a \leq b)$

$$P(X = x) = \frac{1}{b-a+1} \quad \text{for } x = a, a+1, \ldots b,$$

$$var(X) = \frac{(b-a+1)^2 - 1}{12}.$$

$$E(X) = \frac{a+b}{2}.$$

The Poisson Distribution

$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \ldots .$$

How can this arise?

Suppose, on average, in any interval of length 1, λ "blimps" occur. Assume, for pairwise disjoint intervals, these occurrences are independent. In an interval of length t, then, λt blimps occur, on average. What should be the probability that exactly k blimps occur in an interval of length 1?

To get an approximation, cut up the interval into n subintervals of equal length, as above. Then let X_k be the number of blimps that occur in the k^{th} subinterval. If n is large then the probability that X_k exceeds 1 is very small in comparison to the probability that it is 1, so we will simplify the discussion by assuming X_k is either 0 or 1. It is a Bernoulli random variable with $p = \frac{\lambda}{n}$.

Also, $X = X_1 + X_2 + \dots + X_n$ is the number of ~~b~~ blimps in the unit interval. It is binomially distributed with parameters $p = \frac{\lambda}{n}$ and n.

$$P(X=k) = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{1}{k!} \frac{n(n-1)\dots(n-k+1)}{n^k} \lambda^k \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}$$

$$\longrightarrow \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{as } n \to \infty.$$

Poisson Distribution;

Expected Value and Variance

X - Poisson with parameter λ (>0)

$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad (k=0,1,2,\dots)$$

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\right) e^{-\lambda} = e^{\lambda} \cdot e^{-\lambda} = 1.$$

$$E(X) = \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} e^{-\lambda} = \left(\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}\right) \lambda e^{-\lambda} = e^{\lambda} \cdot \lambda \cdot e^{-\lambda} = \lambda.$$

$$E(X(X-1)) = \sum_{k=0}^{\infty} k(k-1) \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} e^{-\lambda} \lambda^2 = \lambda^2.$$

$$\text{var}(X) = E(X^2) - E(X)^2 = E(X(X-1)) + E(X) - E(X)^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda.$$

Expected Values and Variances of some Famous Distributions

Bernoulli, parameter p:

$$E(X) = p \qquad V(X) = pq$$

(More generally) the discrete uniform distribution on $\{a, a+1, \ldots, b\}$:

$$E(X) = \frac{a+b}{2} \qquad V(X) = \frac{(b-a)^2}{12} + \frac{b-a}{6}.$$

Binomial, parameters p and n:

$$E(X) = np \qquad V(X) = npq$$

Geometric, parameter p:

$$E(X) = \frac{1}{p} \qquad V(X) = \frac{q}{p^2}$$

Poisson, parameter λ:

$$E(X) = V(X) = \lambda$$

Hypergeometric, a red marbles, b blue marbles, choose n marbles; X = number of red marbles; $P(X=k) = \dfrac{\binom{a}{k}\binom{b}{n-k}}{\binom{a+b}{n}}$

Put $p = \dfrac{a}{a+b}$.

$E(X) = np$.

$V(X) = npq \dfrac{a+b-n}{a+b-1}$.

(When $a+b$ is large,

$$P(X=k) \approx \binom{n}{k} p^k q^{n-k} \quad)$$

Negative binomial, parameters p, n:

$E(X) = \dfrac{n}{p}$ $\quad V(X) = \dfrac{nq}{p^2}$

Some proofs

for the above

Bernoulli: $P(X=0) = q = 1-p$, $P(X=1) = p$.

$$E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) = p.$$

$$E(X^2) = 0^2\, P(X=0) + 1^2 \cdot P(X=1) = p.$$

$$V(X) = E(X^2) - E(X)^2 = p - p^2 = pq.$$

Discrete uniform distribution on $\{a, a+1, \dots, b\}$.

$$E(X) = \frac{a + (a+1) + \dots + b}{b-a+1} = \frac{a \cdot (b-a+1) + 0 + 1 + \dots + (b-a)}{b-a+1}$$

$$= a + \frac{1}{b-a+1} \cdot \frac{(b-a)(b-a+1)}{2} = \frac{a+b}{2}.$$

$$V(X) = \frac{a^2 + (a+1)^2 + \dots + b^2}{b-a+1} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{0^2 + 1^2 + \dots + (b-a)^2}{b-a+1} - \left(\frac{b-a}{2}\right)^2$$

$$= \frac{1}{b-a+1} \frac{(b-a)(b-a+1)(2(b-a)+1)}{6} - \left(\frac{b-a}{2}\right)^2$$

$$= \frac{b-a}{12}\Big(2 \cdot (2(b-a)+1) - 3(b-a)\Big)$$

$$= \frac{b-a}{12}(b-a+2) = \frac{(b-a)^2}{12} + \frac{b-a}{6}.$$

Binomial distribution, parameters n, p.

It is the sum of n Bernoulli distributions, independent, with parameter p.

Therefore: $E(X) = np$; $V(X) = npq$.

Geometric, parameter p.

k	1	2	3	…	k	…
$P(X=k)$	p	pq	pq^2	…	pq^{k-1}	…

$$E(X) = 1\cdot p + 2\cdot pq + 3\cdot pq^2 + \dots$$

$$= \sum_{k=1}^{\infty} k p q^{k-1} = p \sum_{k=1}^{\infty} k q^{k-1} = \frac{p}{(1-q)^2} = \frac{1}{p}.$$

$$E(X^2) = \sum_{k=1}^{\infty} k^2 p q^{k-1} = p \sum_{k=1}^{\infty} k^2 q^{k-1}$$

$$= p \frac{1+q}{(1-q)^3} = \frac{1+q}{p^2} .$$

$$V(X) = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2} .$$

Homework

Assigned just before the break: 3.30-35, 3.46-49.

New: 3.56, 3.57, 3.59, 3.62, 3.63, 3.66, 3.69, 3.74, 3.76, 3.79, 3.94, 3.110, 3.111, 3.114.