X - a random variable with possible values $x_1, x_2, \dots, x_n$.

Expected value:

$$E(X) = \sum_{k=1}^{n} x_k P(X = x_k),$$

$$Y = g(X).$$

By definition,

$$E(Y) = \sum_{j=1}^{m} y_j P(Y = y_j),$$

where $y_1, \dots, y_m$ are the possible values of Y; i.e., $y_1, \dots, y_m$ are the numbers $g(x_1), \dots, g(x_n)$.

Substitution rule for the expected value of $Y = g(X)$:

$$E(Y) = \sum_{k=1}^{n} g(x_k)\, P(X = x_k).$$

Notice that this reduces to the definition of $E(X)$ when $Y = X$.

Example

Roll a die. $\Omega = \{1, 2, \dots, 6\}$, equally likely. X is the number that comes up. $Y = (X-2)^2$.

THEOREM. $E(cX) = cE(X)$.

Proof. $E(cX) = \sum_i c x_i P(X = x_i)$

$$= c \sum_i x_i P(X = x_i) = c E(X). \quad \square$$

(More long-windedly: cX is a new random variable, $Y = cX$,

$$E(Y) = \sum_{\underset{\substack{\uparrow \\ \text{possible value of } Y}}{y}} y \, P(Y = y).$$

The possible values of Y are the values $c x_i$, where $x_1, \dots, x_n$ are the possible values of X; so the sum is $\sum_{i=1}^{n} c x_i P(X = x_i)$

$= \sim\sim = c E(X)$.)

THEOREM. $E(X+Y) = E(X) + E(Y)$.

Proof.

$$E(X+Y) = \sum_z z \cdot P(X+Y=z)$$

$$= \sum_z \sum_{\substack{i,j:\\ x_i+y_j=z}} (x_i+y_j) P(X=x_i \text{ AND } Y=y_j)$$

$$= \sum_{i,j} (x_i+y_j) P(X=x_i \text{ AND } Y=y_j)$$

$$= \sum_{i,j} x_i P(X=x_i \text{ AND } Y=y_j) + \sum_{i,j} y_j P(X=x_i \text{ AND } Y=y_j)$$

$$= \sum_i x_i \sum_j P(X=x_i \text{ and } Y=y_j) + \sum_j y_j \sum_i P(X=x_i \text{ AND } Y=y_j)$$

$$= \sum_i x_i P(X=x_i) + \sum_j y_j P(Y=y_j)$$

$$= E(X) + E(Y). \qquad \square$$

Variance of X

Given $\mu = E(X)$, the variance of X is defined by:

$$var(X) = E((X-\mu)^2).$$

Clearly $var(X) \geq 0$.

The standard deviation of X is the square root of the variance:

$$\sigma(X) = \sqrt{var(X)}.$$

Homework

3.30 - 35, 3.46 - 49.