Experiment: Flip coin 3 times.

P(exactly 2 heads) ?

$\frac{3}{8}$

(flip n times: 2^n possibilities.)

_ _ _
1 2 3

$2 \cdot 2 \cdot 2 = 2^3$

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

$3 \times \frac{1}{8} = \frac{3}{8} = \frac{\text{number of outcomes in the event}}{\text{number of outcomes in the sample space.}}$

$\Omega = \{HHH, \ldots, TTT\}$

Sample space

$(\Omega = \{0, 1, 2, 3\}\ ?)$

outcomes

$P(E) = \frac{3}{8}$

event, $E = \{HHT, HTH, THH\}$.

$\Omega = \{0, 1, 2, 3\}$

k	$P(k \text{ heads})$
0	$1/8$
1	$3/8$
2	$3/8$
3	$1/8$

Examples using the

Classical (Laplace) model

① Example 1.1 (page 7).

② Tossing two dice. What is the probability that the sum is (_) ?

③ Flipping a coin n times ($n = 1, 2, 3, 4.$) What is the probability of (_) heads ?

Binomial Coefficients and the

Binomial Theorem.

The "Geometric" Model

Experiment: Choose a "random" point in $[0, 10]$. $P(\text{number is} > 8)$?

Another: Choose a random point in the square $[0,1] \times [0,1]$. Call it (x,y).

$P(y \geq x^2)$?

Bernoulli Experiment:

Just 2 possible outcomes.

Interval (say) of possible outcomes

$[0, 1]$

$P($it falls between $\frac{1}{4}$ and $\frac{7}{8}$ feet from top$) = \frac{5}{8} / 1 = \frac{5}{8}$.

$[0, 2]$

For union of finite or countably infinite many intervals ~

$$\frac{\text{total length}}{\text{length of interval } (\Omega)}$$

$P([0, \frac{1}{3}] \cup [\frac{2}{3}, 1]) = \frac{2}{3}$.

Some useful facts from mathematics.

The binomial theorem:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \dots + \binom{n}{n}y^n$$
$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k .$$

Notice that when the expression

$$\underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ factors}} \text{ is simplified}$$

by using the distributive law (but not commutativity of multiplication) until there remain no parentheses, we obtain the sum of all the possible products of n factors each of which is either x or y. For example,

$$(x+y)^3 = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy.$$

The number of terms that have exactly k y's is the coefficient of $x^{n-k}y^k$.

Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \vdots & & & &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

$$
\left.\begin{array}{c}
\binom{n}{0}\ \binom{n}{1} \cdots \binom{n}{k}\ \binom{n}{k+1} \cdots \binom{n}{n} \\
\cdots \quad \binom{n+1}{k+1} \quad \cdots
\end{array}\right) \Longrightarrow
$$

Also, we can describe any one of the terms by saying what positions the y occupies. Number the positions $1, 2, \ldots, n$, the set of positions occupied by y's is a subset of $\{1, 2, \ldots, n\}$. Therefore it is clear that the coefficient of $x^{n-k}y^k$, which we know is $\binom{n}{k}$, is also the number of subsets of $\{1, 2, \ldots, n\}$ (or of any set with n elements) that have exactly k elements.

$\binom{n}{k}$ is the number of k-element subsets of a set with n elements.

"n choose k"

$\{a, b, c, d, e, f\}$
"alphabet"

_ _ _

How many 3-letter words are there?

... with distinct letters?

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

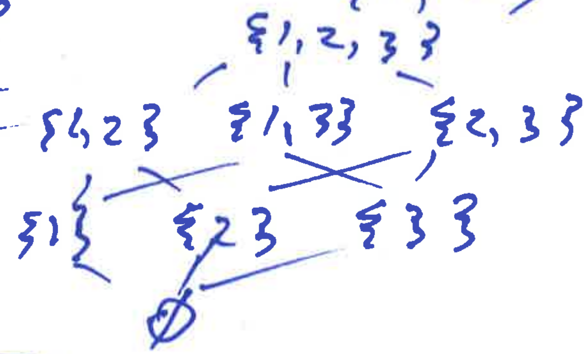
$$= \overset{1\,2\,3}{xxx} + \overset{1\,2\,3}{xxy} + \overset{1\,2\,3}{xyx} + xyy + yxx + yxy + yyx + yyy$$

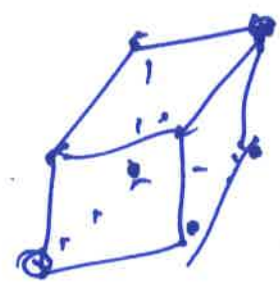
coeff. of xy^2 ?

$$(x+y)(x+y) = xx + yx + xy + yy$$

$$x^3 + 3x^2y + 3xy^2 + y^3$$

$\{2\}$ $\{1,3\}$





$\binom{n}{k}$

$x^{n-k}y^k$

$$\frac{\{1,2,\dots,n\}}{\{1,2,\dots,k\}}$$

Stirling's Approximation

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n .$$

More exactly,

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{4n}\right).$$

An experiment consists of tossing $2n$ coins. What is the probability of exactly n heads?

Answer: $$\binom{2n}{n}\Big/2^{2n} = \frac{(2n)!}{2^{2n}(n!)^2} \approx \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{2^{2n}\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)^2}$$

$$= \frac{\sqrt{4\pi n}\,(2n)^{2n}\,e^{2n}}{\left(\sqrt{2\pi n}\right)^2 e^{2n}\, n^{2n}\, 2^{2n}} = \frac{1}{\sqrt{\pi n}} .$$

n	$n!$	$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$
0	1	0
1	1	$\sqrt{2\pi}/e \approx .922$
2	2	$\sqrt{\pi}\cdot 8/e^2 \approx 1.919$
3	6	$\approx$ 5,836
4	24	$\approx$ 23,506
5	120	$\approx$ 118.019
6	720	$\approx$ 710.078
7	5,040	$\approx$ 4,980
8	40,320	$\approx$ 39,902
9	362,880	$\approx$ 359,537
10	3,628,800	$\approx$ 3,598,700