Some degree 1 examples

① $a_0 = 1,\ a_n = 2a_{n-1}$ when $n \geq 1$.

n	0	1	2	3	4	...	n	...
a_n	1	2	4	8	16	...	2^n	...

$\boxed{\lambda - 2}$

$a_n = 2^n$.

② $b_0 = 1,\ b_n = \frac{1}{2} b_{n-1}$ when $n \geq 1$.

n	0	1	2	3	4	...	n	...
b_n	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	...	$\frac{1}{2^n}$	...

$\boxed{\lambda - \frac{1}{2}}$

③ $b_0 = 5,\ b_n = -b_{n-1}$ when $n \geq 1$.

$\boxed{\lambda + 1}$

n	0	1	2	3	4	...	n	...
b_n	5	-5	5	-5	5	...	$(-1)^n \cdot 5$	...

$\boxed{\text{characteristic polynomials}}$

What can happen?

Do the numbers get bigger as n increases?

Do they get close to 0, or to some other number?

Do they oscillate back and forth?

---

a little Python program

to look at terms of the sequence $\{u_n\}$ given by:

$$u_0 = a$$

$$u_1 = b$$

$$u_n = c \cdot u_{n-1} + d \cdot u_{n-2} \quad \text{for } n \geq 2.$$

Sketch of Algorithm for Solving linear Homogeneous Recurrence Relations

① Look at a few terms of the sequence.

② Determine the characteristic polynomial. Its degree is the same as the degree of the recurrence relation.

③ Find the roots, with multiplicities, and determine the corresponding solutions to the recurrence relation.

④ Write the system of linear equations corresponding to the initial conditions.

⑤ Solve the system for the multipliers.

Some More Examples

(4) $$\boxed{u_0 = 1,\ u_1 = 3,\ u_n = -2u_{n-1} + 15u_{n-2}}$$

(5) $$\boxed{v_0 = 1,\ v_1 = 4,\ v_n = -2v_{n-1} + 15v_{n-2}}$$

(6) $$\boxed{x_0 = 3,\ x_1 = 5,\ x_n = x_{n-1} - x_{n-2}}$$

Some solutions

④ $u_0 = 1,\ u_1 = 3,\ u_n = -2u_{n-1} + 15u_{n-2}$.

n	0	1	2	3
u_n	1	3	9	27

$\lambda^2 + 2\lambda - 15$

$(\lambda+5)(\lambda-3)$

$$\left.\begin{aligned} a_n &= (-5)^n \\ b_n &= 3^n \end{aligned}\right\}$$ two solutions to the recurrence relation

Since 3^n also satisfies the initial conditions, $u_n = b_n = 3^n$ solves our problem.

⑤ $v_0 = 1,\ v_1 = 4,\ v_n = -2v_{n-1} + 15v_{n-2}$.

Again, $(-5)^n$ and 3^n satisfy the recurrence.

$v_n = \alpha\cdot(-5)^n + \beta\cdot(3^n)$

n	0	1	2	3	…
v_n	1	4	7	46	…

$$n=0: \qquad \alpha + \beta = 1$$

$$n=1: \qquad -5\alpha + 3\beta = 4$$

$$8\beta = 9$$

$$-8\alpha = 1$$

$$\alpha = -\frac{1}{8}, \quad \beta = \frac{9}{8}$$

Solution: $v_n = -\frac{1}{8} \cdot (-5)^n + \frac{9}{8} \cdot 3^n$.

What happens as n gets big?

n	0	1	2	3	4	5	6
v_n	1	4	7	46	13	664	-1133

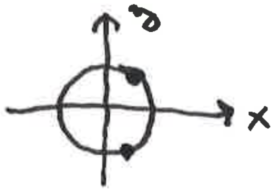
(6) $x_0 = 3$, $x_1 = 5$, $x_n = x_{n-1} - x_{n-2}$

n	0	1	2	3	4	5	6	7
x_n	3	5	2	-3	-5	-2	3	5

The sequence is periodic!!

Characteristic Polynomial: $\lambda^2 - \lambda + 1$

Roots:



$$\lambda = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

$\lambda^6 = 1$, for either of these,

$$\lambda^6 - 1 = (\lambda - 1)(\lambda + 1)(\lambda^2 - \lambda + 1)(\lambda^2 + \lambda + 1).$$

Inhomogeneous Linear Recurrence Relations

$$a_n = 2a_{n-1} + n$$
$$a_0 = 0$$

$$a_n = (\text{linear part}) + (\text{function of } n)$$

---

n	0	1	2	3	4	5 ...
a_n	0	1	4	11	26	57 ...

One solution to the recurrence (that, however, doesn't satisfy the initial condition) is $b_n = -n-2$:

n	0	1	2	3	...
b_n	-2	-3	-4	-5	...

.

Two facts

① If $\{a_n\}$ and $\{b_n\}$ satisfy a nonhomogeneous recurrence relation of the form

$$x_n = (\text{linear part}) + (\text{function of } n)$$

then the difference, $\{c_n = b_n - a_n\}$, satisfies the homogeneous relation: $x_n = (\text{linear part})$.

② If $\{a_n\}$ satisfies a homogeneous linear recurrence relation whose characteristc polynomial is $p(\lambda)$, and $\{b_n\}$ satisfies a homogeneous linear recurrence relation whose characteristic polynomial is $q(\lambda)$, then $\{a_n+b_n\}$ satisfies the linear homogeneous recurrence relation whose characteristic polynomial is the product of the two, namely, $p(\lambda)\cdot q(\lambda)$.

Back to the example:

$$a_n = 2a_{n-1} + n$$

$$a_0 = 0$$

We know a solution to the recurrence: $b_n = -n-2$,

Using ①, we need only find a solution to

$$c_n = 2c_{n-1}$$

for which $b_n + c_n$ satisfies the initial condition.

Since $b_0 = -2$ and a_0 is supposed to be 0, we need $c_0 = 2$.

$$c_n = 2c_{n-1}$$

$$c_0 = 2$$

$$c_n = 2^{n+1}$$

$$a_n = b_n + c_n = -n-2+2^{n+1}.$$

---

n	0	1	2	3
a_n	0	1	4	11

Homework

Problems 1-10 of section 5.3 were assigned before the break.

Here are more:

5.3: 11 (a, b), 12 (a, b), part (a) of 12-15.

For 11 (b), one solution to the recurrence relation is $b_n = \frac{1}{5}$.

For 12 (b), one solution is $b_n = 2 \cdot 8^n$,

(But these don't satisfy the initial conditions.)