Homogeneous Linear Recurrence Relations

(Remember anything about that ??)

Consider the sequence $\{a_n\}_{n=0}^{\infty}$

where $a_0 = 1$

$a_1 = 2$

$a_n = 5a_{n-1} - 4a_{n-2}$ for $n \geq 2$.

We will find a "formula" for the solution, $a_n =$ (something).

First we look for numbers λ for which $\{\lambda^n\}_{n=0}^{\infty}$ is a sequence that satisfies the recurrence relation, so that if $x_n = \lambda^n$ then

$$x_n = 5x_{n-1} - 4x_{n-2} \qquad (n = 2, 3, \ldots).$$

That is, $\lambda^n = 5\lambda^{n-1} - 4\lambda^{n-2}$ $(n \geq 2)$.

As we have seen, it suffices that this works when $n = 2$:

$$\lambda^2 = 5\lambda - 4.$$

characteristic polynomial

We want roots of $\lambda^2 - 5\lambda + 4$.

This factors as $(\lambda - 1)(\lambda - 4)$.

The possible values for λ are 1 and 4. The resulting sequences are

n	0	1	2	3	4 …	n …
$\lambda = 1$	1	1	1	1	1 …	1 …
$\lambda = 4$	1	4	16	64	256 …	4^n …

These sequences both satisfy the recurrence relation, but …

we must find a sequence, satisfying the recurrence relation, and starting off with

n	0	1
a_n	1	2

.

Let us find real numbers α and β for which the sequence $\{\alpha \cdot 1^n + \beta \cdot 4^n\}$ begins this way:

$n=0$: $\alpha + \beta = 1$

$n=1$: $\alpha + 4\beta = 2$.

Then $3\beta = 1$, so $\beta = \frac{1}{3}$ and $\alpha = \frac{2}{3}$.

Define $a_n = \alpha + \beta \cdot 4^n = \frac{2}{3} + \frac{1}{3} \cdot 4^n$.

n	0	1	2	3	...	n	...
$\frac{2}{3}+\frac{1}{3}\cdot 4^n$	1	2	6	22	...	$\frac{2}{3}+\frac{1}{3}4^n$	...

This sequence starts with 1, 2 and and satisfies the recurrence, so $a_n = \frac{2}{3}+\frac{1}{3}\cdot 4^n$ is the solution we sought.

(But we've seen all that.)

Consider $\{b_n\}_{n=0}^{\infty}$, where

$b_0 = 1$, $b_1 = 2$, $b_n = 14 b_{n-1} - 49 b_{n-2}$ $(n \geq 2)$.

We want numbers λ for which

$\lambda^n = 14\lambda^{n-1} - 49\lambda^{n-2}$ $(n = 2, 3, \ldots)$.

For $n = 2$ this is $\lambda^2 = 14\lambda - 49$.

The characteristic polynomial is

$\lambda^2 - 14\lambda + 49$, which factors as

$(\lambda - 7)^2$. The only possibility

for λ is 7.

That only gives one solution.

WE NEED TWO!

7 is a root of $\lambda^2 - 14\lambda + 49 = (\lambda - 7)^2$

with multiplicity 2.

In the earlier example, the characteristic polynomial $\lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4)$ had two roots, each of multiplicity 1.

IMPORTANT FACT: If $p(\lambda)$ is a polynomial of degree d, then $p(\lambda)$ has d roots, when counted with multiplicity.

"d", as in "dog"

another IMPORTANT FACT: If a recurrence relation of degree d has a characteristic polynomial $p(\lambda)$ (of degree d) which has a root λ_0 of multiplicity m then all of the m sequences

$$\{\lambda_0^n\}, \quad \{n\lambda_0^n\}, \{n^2\lambda_0^n\}, \ldots, \{n^{m-1}\lambda_0^n\}$$

satisfy the recurrence relation,

In our case, $p(\lambda) = \lambda^2 - 14\lambda + 49 = (\lambda-7)^2$, and $\lambda_0 = 7$ is a root of multiplicity 2,

According to this fact, both $\{7^n\}$ and $\{n7^n\}$ should be solutions to the recurrence $x_n = 14x_{n-1} - 49x_{n-2}$, Let's verify this.

$x_n = 7^n$ satisfies the recurrence

$x_n = 14\,x_{n-1} - 49\,x_{n-2}$:

$$14 \cdot 7^{n-1} - 49 \cdot 7^{n-2} = 7^{n-2}(14 \cdot 7 - 49)$$

$$= 7^{n-2}(98 - 49) = 7^{n-2} \cdot 49$$

$$= 7^{n-2} \cdot 7^2 = 7^n.$$

$x_n = n\,7^n$ satisfies the recurrence, also:

$$14 \cdot (n-1) \cdot 7^{n-1} - 49 \cdot (n-2) \cdot 7^{n-2}$$

$$= 7^{n-2}\big(14 \cdot (n-1) \cdot 7 - 49 \cdot (n-2)\big)$$

$$= 7^{n-2}\big((98n - 98) - (49n - 98)\big)$$

$$= 7^{n-2} \cdot 49n = n \cdot 7^n.$$

(Continuing with the example -)

$b_0 = 1$, $b_1 = 2$, $b_n = 14\, b_{n-1} - 49\, b_{n-2}$ $(n \geq 2)$.

What is b_n ?

The characteristic polynomial is

$\lambda^2 - 14\lambda + 49$.

It has one root (of multiplicity 2):

namely, 7.

The recurrence has the two solutions

$\{7^n\}_{n=0}^{\infty}$ and $\{n \cdot 7^n\}_{n=0}^{\infty}$.

We must find suitable α and β,

yielding the solution b_n of the form

$$b_n = 7^n \cdot \alpha + n \cdot 7^n \cdot \beta.$$

For the first two values (when $n = 0$ and 1) we get

$$7^0 \cdot \alpha + 0 \cdot 7^0 \cdot \beta = b_0 = 1$$

$$7^1 \cdot \alpha + 1 \cdot 7^1 \cdot \beta = b_1 = 2.$$

That is, we need to solve the two linear equations in two unknowns:

$$\begin{aligned} \alpha \qquad\quad &= 1 \\ 7\alpha + 7\beta &= 2. \end{aligned}$$

The solution: to this system:

$$\alpha = 1$$
$$\beta = -\frac{5}{7}.$$

Then $b_n = 7^n - \frac{5}{7} \cdot 7^n \cdot n$

$\qquad = (7 - 5n) \cdot 7^{n-1},$

n	0	1	2	...
$(7-5n) \cdot 7^{n-1}$	1	2	$-3 \cdot 7 = -21$	...

The roots might not be real numbers.

(What then?)

A near-trivial example (?)

$$\boxed{c_0 = 7, \quad c_1 = 5, \quad c_n = -c_{n-2} \ (n \geq 2).}$$

We want the numbers λ for which $\{\lambda^n\}$ is a solution to the recurrence:

$$\lambda^n = -\lambda^{n-2}.$$

The characteristic polynomial: $\lambda^2 + 1$.

Of course the solution is pretty simple:

n	0	1	2	3	4	5	6	...
c_n	7	5	-7	-5	7	5	-7	...

What's the formula for c_n?

The roots of the characteristic polynomial are i and $-i$.

We want

$\alpha \cdot (i)^n + \beta (-i)^n$ to be c_n.

$n=0$: $\alpha + \beta = 7$

$n=1$: $\alpha i - \beta i = 5$

$$\alpha + \beta = 7$$
$$-\alpha + \beta = 5i$$

$$2\beta = 7+5i$$
$$2\alpha = 7-5i$$

$$\alpha = \frac{7-5i}{2}$$
$$\beta = \frac{7+5i}{2}$$

$$\boxed{c_n = \frac{7-5i}{2} \cdot i^n + \frac{7+5i}{2} \cdot (-i)^n.}$$

the Fibonacci numbers

0, 1, 1, 2, 3, 5, 8, 13, ...

$f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ $(n \geq 2)$.

Characteristic equation: $\lambda^2 = \lambda + 1$

Characteristic polynomial: $\lambda^2 - \lambda - 1$.

Roots (by the Quadratic formula):

$$\lambda = \frac{1 \pm \sqrt{5}}{2}.$$

Solution:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

(See Problem 22 on pages 171 and 172 in the book if you want more details.)

$$\boxed{g_0 = g_1 = 1, \quad g_n = \frac{6}{5} g_{n-1} - g_{n-2} \quad (n \geq 2).}$$

Char. poly. – $\lambda^2 - \frac{6}{5}\lambda + 1 = 0.$

Roots – $\frac{3}{5} + \frac{4}{5} i, \quad \frac{3}{5} - \frac{4}{5} i$