Recurrence Relations

$a_1 = 1, \quad a_n = 2a_{n-1}$ when $n \geq 2$.

$a_n = 2^{n-1}$

k	1	2	3	4 ...	k ...
a_k	1	2	4	8 ...	2^{k-1} ...

$b_1 = 1, \; b_2 = 2, \; b_n = 2b_{n-1} - b_{n-2}$

when $n \geq 3$.

$b_n = n$

k	1	2	3	4 ...	k ...
b_k	1	2	3	4 ...	k ...

$b_k = k$, for all $k \in \mathbb{N}$.

Proof. Since $b_1 = 1$ and $b_2 = 2$, the equation is true when $k=1$ or $k=2$.

Suppose $b_{k-2} = k-2$ and $b_{k-1} = k-1$.

Then
$$\begin{aligned} b_k &= 2b_{k-1} - b_{k-2} \\ &= 2(k-1) - (k-2) \\ &= 2k-2-k+2 = k. \end{aligned}$$

By PMI, $b_k = k$ for all $k \in \mathbb{N}$. $\square$

Linear Recurrence Relations

$$a_n = \alpha_1 a_{n-1} + \alpha_2 a_{n-2} + \dots + \alpha_m a_{n-m} + \beta$$

of degree m

$\beta = 0$

homogeneous

$$a_n = 2a_{n-1}$$

degree 1

$$b_n = b_{n-1}$$

degree 2 $\rightarrow$ $g_n = g_{n-2}$

k	1	2	3	4	5	6
g_k	2	5	2	5	2	5 ...

$P(n)$.

Proof. $P(1)$. ✓

If $n \geq 1$ then

$P(n) \Rightarrow P(n+1)$.

By PMI, ✓✓.

$S(1), S(2), \ldots, S(k), \ldots$

~~P(n) and P~~

$P(n)$: $S(n)$ and $S(n+1)$.

$P(1)$ — $S(1)$ and $S(2)$.

$P(n) \to P(n+1)$ — $S(n)$ and $S(n+1)$

$\to$ $\underline{S(n+1)}$ and $\underline{S(n+2)}$

$$a_n = a_{n-1} - a_{n-2}$$

n	1	2	3	4	5	6	7	8	9
a_n	2	5	3	-2	-5	-3	2	5	3

$s_0 = 1,\quad s_1 = 1,$

$$s_n = 7s_{n-1} - 12s_{n-2}$$

n	0	1	2	3	4	...	k	...
s_n	1	1	-5	-47	-269	...		

$$\begin{array}{r} 47 \\ 7 \\ \hline -329 \\ +60 \\ \hline -269 \end{array}$$

$t_0 = 1,\ t_1 = 3,\ t_n = 7t_{n-1} - 12t_{n-2}$

n	0	1	2	3	4	...	k	...
t_n	1	3	9	27	81		3^k	

$$\begin{array}{r} 63 \\ 36 \\ \hline \end{array}$$

$t_n = 3^n$?

$t_n = 3^n$.

Proof. It works for $n = 0$ and 1.

Suppose $n \geq 0$ and $t_n = 3^n$ and $t_{n+1} = 3^{n+1}$. Then

$$t_{n+2} = 7\,t_{n+1} - 12\,t_n$$
$$= 7 \cdot 3^{n+1} - 12 \cdot 3^n$$
$$= 3^n \cdot (7 \cdot 3 - 12 \cdot 1)$$
$$= 3^n \cdot 9 = 3^{n+2}.$$

So it works for $n+2$.

By PMI, $t_n = 3^n$ for all $n \in \mathbb{N}$. $\square$

$$y_n = 7y_{n-1} - 12y_{n-2}.$$

n	0	1	2	3	$\ldots$
y_n	$\underline{1}$	$\underline{\lambda}$	λ^2	λ^3	$\ldots$

$$y_2 = 7y_1 - 12y_0$$

$$\rightarrow \quad \lambda^2 = 7\lambda - 12$$

$$\lambda^2 - 7\lambda + 12 = 0$$

$$(\lambda - 3)(\lambda - 4) = 0$$

$$\cancel{\lambda^4 = 7\lambda^3 - 12\lambda}$$

$$\lambda^3 = 7\lambda^2 - 12\lambda$$

$$\lambda^4 = 7\lambda^3 - 12\lambda^2$$

$$\vdots$$

$$a_n = 3^n \qquad X_n = 7X_{n-1} - 12X_{n-2}$$

$$b_n = 4^n$$

$$c_n = a_n + b_n = 3^n + 4^n$$

$$\cancel{c_n =}\ 7c_{n-1} - 12c_{n-2}$$

$$= 7(a_{n-1} + b_{n-1}) - 12(a_{n-2} + b_{n-2})$$

$$= (7a_{n-1} - 12a_{n-2}) + (7b_{n-1} - 12b_{n-2})$$

$$= \quad a_n \quad + \quad b_n = c_n$$

$$7 \cdot 3^n \qquad\qquad 7 \cdot a_n$$

$$s_n = \alpha \cdot 3^n + \beta \cdot 4^n$$

$$\left.\begin{aligned} 1 = S_0 &= \alpha + \beta \\ 1 = S_1 &= 3\alpha + 4\beta \end{aligned}\right\}$$

$$\begin{array}{rrl} -3 & \alpha + \beta & = 1 \\ + & 3\alpha + 4\beta & = 1 \\ \hline & \beta & = -2 \\ & \alpha & = 3 \end{array}$$

$$\boxed{S_n = 3 \cdot 3^n - 2 \cdot 4^n}$$

n	0	1	2	3	4
S_n	1	1	-5		

$y_0 = 1, \quad y_1 = 2$

$y_n = 7y_{n-1} - 10y_{n-2}$

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n	0	1	2	3	$\dots$	k	$\dots$
y_n	1	2	4	8	$\dots$	$\underline{2^k}$	$\dots$

$\lambda^2 = 7\lambda - 10$

$\rightarrow \quad \lambda^2 - 7\lambda + 10 \quad \Leftarrow$ char. poly.

$= (\lambda - 2)(\lambda - 5) \Leftarrow$

$$w_0 = w_1 = 1$$
$$w_n = 7w_{n-1} - 10w_{n-2}$$

n	0	1	2	3	
w_n	1	1	-3	-31	
	1	2	4	8	α
	1	5	25	125	β

$$\begin{aligned} -2: \quad & \alpha + \beta = 1 \\ 1: \quad & 2\alpha + 5\beta = 1 \\ \hline & 3\beta = -1 \end{aligned}$$

$\beta = -\frac{1}{3}$

$\alpha = \frac{4}{3}.$

$$w_n = \frac{4}{3} \cdot 2^n - \frac{1}{3} \cdot 5^n$$

Homework

5.3: 1 - 10.