1. (5.1, # 14) For any square matrix A, prove that A and A^t have the same characteristic polynomial.

Proof:

Let $B := A - xI$ and $C := A^t - xI$ (use x to denote our variable since t denotes the transpose in this problem). Then $B^t = (A - xI)^t$. We next show that for arbitrary matrices X and Y (of the same size), $(X+Y)^t = X^t + Y^t$. Clearly $(X^t + Y^t)_{ij} = X_{ji} + Y_{ji} = [(X+Y)^t]_{ji}$, where X_{ij} is the entry in the ith row and jth column of the matrix X. Hence the claim is proved. Thus $(A - tI)^t = A^t - (tI)^t = A^t - xI = C$. Hence $\mathrm{char}(A) = \det(B)$ which by an earlier result equals $\det(B^t) = \det(C) = \mathrm{char}(A^t)$.

2. (Sec. 5.2, # 18a) Prove that if T and U are simultaneously diagonalizable linear operators on V, then T and U commute (i.e., $T \circ U = U \circ T$).

Proof: First we note that if C and D are both diagonal matrices, then $CD = DC$. To see this let

$$C = \begin{pmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & c_n \end{pmatrix} \text{ and } D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & d_n \end{pmatrix} \text{ then,}$$

$$CD = \begin{pmatrix} c_1 d_1 & 0 & \cdots & 0 \\ 0 & c_2 d_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & c_n d_n \end{pmatrix} = DC$$

Now suppose that β is a basis for V such that both $[T]_\beta$ and $[U]_\beta$ are diagonal. Then $[T \circ U]_\beta = [T]_\beta [U]_\beta$ and since these are diagonal matrices, they commute. Thus $[T]_\beta [U]_\beta = [U]_\beta [T]_\beta = [U \circ T]_\beta$. Since $T \circ U$ and $U \circ T$ have the same matrix with respect to β, they must be the same transformation. Done.

3. (Sec. 5.4, # 13) Let A be an $n \times n$ matrix. Prove that

$$\dim(\text{span}\{I_n, A, A^2, \ldots\}) \leq n.$$

Proof: By Caley-Hamilton, A is a root of the characteristic polynomial of A, which has degree n. Thus there exists scalars, $a_0, a_1, \ldots a_{n-1}$ such that

$$A^n = a_0 I_n + a_1 A + \cdots + a_{n-1} A^{n-1}.^*$$

In other words A^n is a linear combination of the set $\{A^i\}_{i=0,\ldots n-1}$. Then

$$A^{n+1} = A(A^n) = A(a_0 I_n + a_1 A + \cdots + a_{n-1} A^{n-1}) = a_0 A + \cdots + a_{n-1} A^n.$$

Then, since A^n can be written in terms of lower powers of A, it follows that A^{n+1} is in the span of $\{I, A, \ldots, A^{n-1}\}$. Clearly we can repeat this argument (by induction) for all A^k, where $k > n$. Thus the set $\{I, A, \ldots, A^{n-1}\}$ is a spanning set for the space span$\{I_n, A, A^2, \ldots\}$. Since this spanning set only has n elements, the dimension of this space is at most n.