(1) (6.16) Prove that the mapping from $U(16)$ to itself given by $x \mapsto x^3$ is an isomorphism.

Proof: Note that $U(16) = \{1, 3, 5, 7, 9, 11, 13, 15\}$. First we show that the map is a bijection. Since the group is a finite set, it suffices to show that it is an injection. One could do this by brute force, but we give an alternate proof (which still uses brute force). Suppose that $x^3 = y^3$, for some $x, y \in U(16)$. Then $(xy^{-1})^3 = 1$. Thus xy^{-1} either has order 1 or 3. However, one can check that every element of $U(16)$ has order 1, 2 or 4 (we could have used Lagrange here, if we knew about it). Hence xy^{-1} has order 1, or $xy^{-1} = 1$. Therefore $x = y$, which proves that the map is an injection. Thus the map is a bijection. Next we show that the map is structure preserving. But $(xy)^3 = x^3y^3$. Done!

(2) (6.22) Prove or disprove that $U(20)$ and $U(24)$ are isomorphic.

Proof: We use some more brute force.

$$U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}.$$

Then $3^2 = 9, 3^3 = 7, 3^4 = 1$. In particular $|3| = 4$.

$$U(24) = \{1, 5, 7, 11, 13, 17, 19, 23\}.$$

Then $5^2 = 1, 7^2 = 49 = 1, 11^2 = 121 = 1, 13^2 = 169 = 1, 17^2 = 289 = 1, 19^2 = 361 = 1$. Hence for all $a \in U(24)$, either $|a| = 1$ or $|a| = 2$. In particular, $U(24)$ does not have an element of order 4. Since $U(20)$ does have an element of order 4, the groups cannot be isomorphic.

(3) (6.24) Show that the groups $G = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$ and $H = \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} | a, b \in \mathbb{Q} \right\}$ (under addition) are isomorphic.

Proof: We define a map $\phi : G \to H$ via

$$\phi(a + b\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$$

Given the definition of G and H, it is clear that ϕ is a bijection between the two sets. Next we check that it preserves the structure:

$$\phi(a + b\sqrt{2}) + \phi(c + d\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} + \begin{pmatrix} c & 2d \\ d & c \end{pmatrix} =$$

$$\begin{pmatrix} a + c & 2(b + d) \\ b + d & a + c \end{pmatrix} = \phi(a + c + (b + d)\sqrt{2}) = \phi(a + b\sqrt{2} + c + d\sqrt{2})$$

(4) (7.8) Suppose that $a \in G$ has order 15. Find all the left cosets of $H = \langle a^5 \rangle$ in $\langle a \rangle$.

Proof: We list them:

$$\begin{aligned} eH = H &= \{e, a^5, a^{10}\} \\ aH &= \{a, a^6, a^{11}\} \\ a^2H &= \{a^2, a^7, a^{12}\} \\ a^3H &= \{a^3, a^8, a^{13}\} \\ a^4H &= \{a^4, a^9, a^{14}\} \end{aligned}$$

The above list includes all elements of $\langle a \rangle$. Hence all left cosets are on the list.