Krull Dimension and Going-Down in Fixed Rings

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Basics

$R$ will always be a commutative ring and $G$ a group of (ring) automorphisms of $R$.

We let $R^G$ denote the fixed ring, that is,

$$R^G = \{ a \in R : g(a) = a \text{ for all } g \in G \}.$$

Thus $R^G$ is a subring of $R$. 

Example

Let $R = K[x, y]$; let $G = \langle g \rangle$ where $g : R \rightarrow R$ via $g(x) = -x$, $g(y) = -y$ and $g$ fixes the elements of $K$. Then $R^G = K[x^2, xy, y^2]$. 

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Types of Questions

1. What properties of $R$ are inherited by the ring $R^G$?

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Hilbert’s $XIV^{th}$ Problem

Let $K$ be a field, $x_1, x_2, \ldots, x_n$ algebraically independent elements over $K$, and $G$ a subgroup of $GL(n, K)$. Is the fixed ring $K[x_1, x_2, \ldots, x_n]^G$ (or ring of invariants) finitely generated over $K$?
Solution

There were some partial positive answers by Zariski and Noether. However in 1958 Nagata showed that the answer in general was no.
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Definition

The group $G$ is said to be locally finite on $R$ if for each $a \in R$, the orbit of $a$ under the action of $G$ is finite, i.e., for each $a \in R$, the set \[ \{ g(a) : g \in G \} \] is finite.
Proposition

If $G$ is locally finite, then $R^G \subset R$ is an integral extension. That is, every element of $R$ satisfies a monic polynomial over $R^G$. 

Proof

Let $a \in R$, then $a$ satisfies the polynomial $f(x) = \prod_{b \in O(a)} (x - b)$, where $O(a)$ denotes the orbit of $a$ under the action of $G$. The coefficients are in $R^G$. 

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where $\mathcal{O}(a)$ denotes the orbit of $a$ under the action of $G$. The coefficients are in $R^G$. 
Type 1 Question:

Does $R$ Noetherian imply $R^G$ Noetherian under the assumption that $G$ is finite?

Nagarajan (1968) constructed an example of a Noetherian ring $R$ ($R = \mathbb{F}[[x,y]]$) of characteristic 2, and a group $G$ of order 2 acting on $R$ such that $R^G$ was not Noetherian.

Bergman (1971) showed that if the order of $G$ was a unit of $R$, then $R$ Noetherian implies that $R^G$ is Noetherian.
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Answers
1. Nagarajan (1968) constructed an example of a Noetherian ring $R$ ($R = F[[x, y]]$) of characteristic 2, and a group $G$ of order 2 acting on $R$ such that $R^G$ was not Noetherian.
2. Bergman (1971) showed that if the order of $G$ was a unit of $R$, then $R$ Noetherian implies that $R^G$ is Noetherian.
A Question We Considered

When does $R$ Artinian imply that $R^G$ Artinian?
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Definitions

1. Recall that $R$ is *Artinian* if $R$ has the descending chain condition on ideals.

2. The *Krull dimension* of $R$ (which we denote $\text{dim}(R)$) is the length of the longest chain of prime ideals.
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Definitions

1. Recall that $R$ is **Artinian** if $R$ has the descending chain condition on ideals.
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Basic Facts

1. Artinian $\iff$ Noetherian & $\text{dim}(R) = 0$. Also note that $R$ Artinian implies that $R$ has only finitely many maximal ideals.
2. If $G$ is locally finite, then $R$ is integral over $R^G$. This in turn implies that $\text{dim}(R^G) = \text{dim}(R)$. 
Transfer of Krull Dimension

1. Without any assumptions on $G$ we have examples such that $\dim(R) - \dim(R^G)$ is any integer we want (positive or negative). We can even have $\dim(R) = \infty$ and $\dim(R^G) = 0$.

2. On the other hand we have no examples where the dimensions differ if $\dim(R) = 0$. 

(Note: No assumptions on $G$.)

Theorem
If $\dim(R) = 0$ (so all prime ideals are maximal) and $R$ has $n$ maximal ideals, then $R^G$ has at most $n$ maximal ideals and $\dim(R^G) = 0$. 

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If $\dim(R) = 0$ (so all prime ideals are maximal) and $R$ has $n$ maximal ideals, then $R^G$ has at most $n$ maximal ideals and $\dim(R^G) = 0$. 
Corollary

If $R$ is Artinian, then $\dim(R^G) = 0$ and $R^G$ has finitely many maximal ideals.

We saw how to modify Nagarajan ($R$ Noetherian, $G$ finite, yet $R^G$ not Noetherian) to provide an example of $R$ Artinian and $G$ finite, yet $R^G$ is not Artinian.
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Let $F := \mathbb{Z}_2(a_i, b_i, i \geq 1)$, where the $a_i, b_i$ are commuting algebraically independent indeterminates over the field $\mathbb{Z}_2$ with two elements. (Note $F$ is the field of quotients of the integral domain $\mathbb{Z}_2[a_i, b_i]$.)

Consider the formal power series ring $S := F[[X, Y]]$. Then $S$ has an automorphism $g$ given

$$g(X) := X, \quad g(Y) := Y$$

and

$$g(a_i) := a_i + p_{i+1}Y, \quad g(b_i) := b_i + p_{i+1}X,$$

where $p_i := a_iX + b_iY$. Let $G := \langle g \rangle$. Since $g^2$ is the identity map, $|G| = 2$. It is well known that $S$ is a Noetherian ring. However, Nagarajan has shown that $R^G$ is not a Noetherian ring.
Consider the ideal $J := (X^2, Y^2)$ of $S$. Since $X^2$ and $Y^2$ are each fixed by $g$, it is easy to see that $J$ is $G$-invariant, and so $G$ acts (via ring automorphisms) on $R := S/J$. Of course, $R$ inherits the property of being a Noetherian ring from $S$. Moreover, it is easy to check that $R$ is zero-dimensional and local. Thus $R$ is an Artinian ring, yet we can show that $S^G$ is not Noetherian, hence not Artinian.

The proof does not involve describing all the elements of $R^G$ explicitly (that seems too difficult). Rather, one shows that a certain family of elements are in $R^G$, from which we are able to create a strictly ascending chain of ideals.
Basic Ideas of Proof

Let $x$ and $y$ denote the canonical images of $X$ and $Y$, respectively, in $R$. A degree argument shows that the set $\{1, x, y, xy\}$ is a basis of the vector space $R$ over the field $F$. Also, note that when $a_i$ is viewed as an element of $R$, then

$$g(a_i) = a_i + a_{i+1}xy \quad (\text{since } y^2 = 0 \in R).$$

Thus

$$g(a_iy) = g(a_i)g(y) = (a_i + a_{i+1}xy)y = a_iy,$$

and so $a_iy \in R^G$. We show that the sequence of ideals of $R^G$ given in

$$ (a_1y) \subseteq (a_1y, a_2y) \subseteq (a_1y, a_2y, a_3y) \subseteq \ldots $$

is strictly ascending.
Recall that if \( G \) is locally finite on \( R \), then \( R \) is integral over \( R_G \). Note since \( R_G \subset R \), there is a map (called the contraction map) \( \text{Spec}(R) \rightarrow \text{Spec}(R_G) \), given by \( P \mapsto P \cap R_G \).
Recall

If $G$ is locally finite on $R$, then $R$ is integral over $R^G$.

Note since $R^G \subset R$, there is a map (called the contraction map) $\text{Spec}(R) \to \text{Spec}(R^G)$, given by $P \mapsto P \cap R^G$. 
Integrality

Integrality has a number of consequences for this map.

For example:
1. The map is onto.

2. If $P \subset Q$ are elements of Spec$(R)$, then $P \cap R^G \subset Q \cap R^G$.

3. “Going-up” (GU). If $p \subset q$ are elements of Spec$(R^G)$ and $P \in$ Spec$(R)$ such that $P \cap R^G = p$, then there exists $Q \in$ Spec$(R)$ such that $P \subset Q$ and $Q \cap R^G = q$. In other words the diagram can be filled in:

$$
\begin{array}{c}
P \subset ? \\
\downarrow \quad \downarrow \\
p \subset q
\end{array}
$$
Definition of a Related Property

A ring injection $S \hookrightarrow R$ satisfies going-down (GD) if the following diagram can be filled in:

$$
\begin{array}{c}
? \\
\downarrow \\
p
\end{array}
\subset
\begin{array}{c}
Q \\
\downarrow \\
q
\end{array}
$$

Integral extensions do not have to satisfy GD. Nonetheless we were able to show a stronger property for fixed rings. First a definition.

Definition of a Stronger Version of GD

An inclusion map of rings $S \hookrightarrow R$ is said to be \textit{universally going-down}, if for any commutative $R$-algebra $T$, the canonical map $T \rightarrow T \otimes_S R$ satisfies going-down.

$S \hookrightarrow R$ satisfies universally going-down if and only if the canonical map $S[x_1, x_2, \ldots, x_n] \hookrightarrow R[x_1, x_2, \ldots, x_n]$ satisfies going-down for each $n$. 
Note that if $G$ acts on $R$, then this action naturally extends to $R[x_1, x_2, \ldots, x_n]$.

**Theorem**

If $G$ is locally finite on $R$, then $R^G \hookrightarrow R$ is universally going-down.

We did not do this at one time.

**Steps in Proof**

We first proved this when $G$ is locally finite and $R$ Noetherian (what we really needed was that there are only finitely many primes minimal over an arbitrary ideal).
Then we realized, using a criteria of Kaplansky, that the obstruction to going-down was a finite data set. Basically, if the inclusion $R^G$ into $R$ does not satisfy universally going-down, then there exists finitely many elements $a_1, a_2, \ldots, a_n \in R$ that screw things up. Take these elements and their orbits (which still leave you with a finite set) and take the ring generated by $\mathbb{Z}$ and these finite number of elements. This ring, call it $T$, is Noetherian, and $G$ still acts on this ring. By construction $T^G \hookrightarrow T$ does not satisfies universally going-down. But by earlier result it does - a contradiction.

For a while we did not have an example to show that if we dropped the locally finite assumption, then $R^G \hookrightarrow R$ does not satisfy GD. But finally we did come up with an example using a semigroup ring.
Outline of the Construction

- We construct an abelian semigroup $S$ (under $+$) via generators and relations.
- We define an automorphism group $G$ acting on $S$.
- We let $R = K[S] = K[x^s : s \in S]$, where $K$ is a field. The action of $G$ extends in a natural fashion to $R$.
- We show that $R^G = K[S^G]$.
- We construct the appropriate diagram and show that it cannot be completed.
The Semigroup

Let $M$ denote the free abelian monoid on the symbols $\{A, C_i | i \in \mathbb{Z}\}$, written additively.

We define a congruence relation on this monoid, as follows. Let $nA + C_{i_1} + C_{i_2} + \cdots + C_{i_k}$ and $mA + C_{j_1} + \cdots + C_{j_t}$ be arbitrary elements of $M$, where $t$ is a nonnegative integer, the $C_k$ are not necessarily distinct, and $n$ and $m$ are nonnegative integers. Then we declare that

$$nA + C_{i_1} + C_{i_2} + \cdots + C_{i_k} \equiv mA + C_{j_1} + \cdots + C_{j_t}$$

if either $nA + C_{i_1} + C_{i_2} + \cdots + C_{i_k} = mA + C_{j_1} + \cdots + C_{j_t}$ or $[n = m \neq 0$ and $k = t]$. This is a congruence relation on $M$ (that is, an equivalence relation on $M$ that is compatible with the operation of addition on $M$). Let $S$ denote the factor semigroup $M/\equiv$. 
The Automorphism Group of $S$

First we define $G$ on $M$. Let $g : M \to M$ be given by $g(A) = A$ and $g(C_i) = C_{i+1}$ and then extending linearly. It is clear that if $x, y \in M$ satisfy $x \equiv y$, then $g(x) \equiv g(y)$. Hence, $g$ induces an automorphism of $S$, also denoted $g$. Then $g$ has infinite order and $G = \langle g \rangle$ acts on $S$.

The Example

With $R = K[S]$, we have that $K[S^G] = R^G \subset R$ does not satisfy going-down (much less universally going-down).

The End

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