Krull Dimension and Going-Down in Fixed Rings

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April 19, 2006

Basics

R will always be a commutative ring and G a group of (ring) automorphisms of R.

We let R^G denote the fixed ring, that is,

$$R^G = \{a \in R : g(a) = a \text{ for all } g \in G\}.$$

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Example

Let $R = K[x, y]$; let $G = \langle g \rangle$ where $g : R \to R$ via $g(x) = -x$, $g(y) = -y$ and g fixes the elements of K. Then

$$R^G = K[x^2, xy, y^2]$$

Types of Questions

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2. What is the relation between R and the subring R^G?

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Hilbert's XIV^{th} Problem

Let K be a field, $x_1, x_2, \ldots, x_n$ algebraically independent elements over K, and G a subgroup of $GL(n, K)$. Is the fixed ring $K[x_1, x_2, \ldots, x_n]^G$ (or ring of invariants) finitely generated over K?

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Definition

The group G is said to be locally finite on R if for each $a \in R$, the orbit of a under the action of G is finite, i.e., for each $a \in R$, the set $\{g(a) : g \in G\}$ is finite.

Proposition

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Proof

Let $a \in R$, then a satisfies the polynomial

$$f(x) = \Pi_{b \in \mathcal{O}(a)}(x - b),$$

where $\mathcal{O}(a)$ denotes the orbit of a under the action of G. The coefficients are in R^G.

Type 1 Question:

Does R Noetherian imply R^G Noetherian under the assumption that G is finite?

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Answers

1. Nagarajan (1968) constructed an example of a Noetherian ring R ($R = F[[x, y]]$) of characteristic 2, and a group G of order 2 acting on R such that R^G was not Noetherian.
2. Bergman (1971) showed that if the order of G was a unit of R, then R Noetherian implies that R^G is Noetherian.

A Question We Considered

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Basic Facts

1. Artinian $\Leftrightarrow$ Noetherian & $\dim(R) = 0$. Also note that R Artinian implies that R has only finitely many maximal ideals.
2. If G is locally finite, then R is integral over R^G. This in turn implies that $\dim(R^G) = \dim(R)$.

Transfer of Krull Dimension

1. Without any assumptions on G we have examples such that $\dim(R) - \dim(R^G)$ is any integer we want (positive or negative). We can even have $\dim(R) = \infty$ and $\dim(R^G) = 0$
2. On the other hand we have no examples where the dimensions differ if $\dim(R) = 0$.

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However with an assumption on R we have the following (Note: No assumptions on G.):

Theorem

If $\dim(R) = 0$ (so all prime ideals are maximal) and R has n maximal ideals, then R^G has at most n maximal ideals and $\dim(R^G) = 0$.

Corollary

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We saw how to modify Nagarajan (R Noetherian, G finite, yet R^G not Noetherian) to provide an example of R Artinian and G finite, yet R^G is not Artinian.

The Construction of Nagarajan

Let $F := \mathbb{Z}_2(a_i, b_i, i \geq 1)$, where the a_i, b_i are commuting algebraically independent indeterminates over the field $\mathbb{Z}_2$ with two elements. (Note F is the field of quotients of the integral domain $\mathbb{Z}_2[a_i, b_i]$.)

Consider the formal power series ring $S := F[[X, Y]]$. Then S has an automorphism g given

$$g(X) := X,\ g(Y) := Y \text{ and}$$

$$g(a_i) := a_i + p_{i+1}Y, \quad g(b_i) := b_i + p_{i+1}X,$$

where $p_i := a_iX + b_iY$. Let $G := \langle g \rangle$. Since g^2 is the identity map, $|G| = 2$. It is well known that S is a Noetherian ring. However, Nagarajan has shown that R^G is not a Noetherian ring.

Our Variation

Consider the ideal $J := (X^2, Y^2)$ of S. Since X^2 and Y^2 are each fixed by g, it is easy to see that J is G-invariant, and so G acts (via ring automorphisms) on $R := S/J$. Of course, R inherits the property of being a Noetherian ring from S. Moreover, it is easy to check that R is zero-dimensional and local. Thus R is an Artinian ring, yet we can show that S^G is not Noetherian, hence not Artinian.

The proof does not involve describing all the elements of R^G explicitly (that seems too difficult). Rather, one shows that a certain family of elements are in R^G, from which we are able to create a strictly ascending chain of ideals.

Basic Ideas of Proof

Let x and y denote the canonical images of X and Y, respectively, in R. A degree argument shows that the set $\{1, x, y, xy\}$ is a basis of the vector space R over the field F. Also, note that when a_i is viewed as an element of R, then

$$g(a_i) = a_i + a_{i+1}xy \text{ (since } y^2 = 0 \in R).$$

Thus

$$g(a_i y) = g(a_i)g(y) = (a_i + a_{i+1}xy)y = a_i y,$$

and so $a_i y \in R^G$. We show that the sequence of ideals of R^G given in

$$(a_1 y) \subseteq (a_1 y, a_2 y) \subseteq (a_1 y, a_2 y, a_3 y) \subseteq \ldots$$

is strictly ascending.

A Question of Type 2

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Recall

If G is locally finite on R, then R is integral over R^G.

Note since $R^G \subset R$, there is a map (called the *contraction map*) $\mathrm{Spec}(R) \to \mathrm{Spec}(R^G)$, given by $P \mapsto P \cap R^G$.

Integrality

Integrality has a number of consequences for this map.

For example:

1. The map is onto.

2. If $P \subset Q$ are elements of $\mathrm{Spec}(R)$, then $P \cap R^G \subset Q \cap R^G$.

3. "Going-up" (GU). If $p \subset q$ are elements of $\mathrm{Spec}(R^G)$ and $P \in \mathrm{Spec}(R)$ such that $P \cap R^G = p$, then there exists $Q \in \mathrm{Spec}(R)$ such that $P \subset Q$ and $Q \cap R^G = q$. In other words the diagram can be filled in:

$$\begin{array}{ccc} P & \subset & ? \\ \downarrow & & \downarrow \\ p & \subset & q \end{array}$$

Definition of a Related Property

A ring injection $S \hookrightarrow R$ satisfies going-down (GD) if the following diagram can be filled in:

$$\begin{array}{ccc} ? & \subset & Q \\ \downarrow & & \downarrow \\ p & \subset & q \end{array}$$

Integral extensions do not have to satisfy GD. Nonetheless we were able to show a stronger property for fixed rings. First a definition.

Definition of a Stronger Version of GD

An inclusion map of rings $S \hookrightarrow R$ is said to be *universally going-down*, if for any commutative R-algebra T, the canonical map $T \to T \otimes_S R$ satisfies going-down.

$S \hookrightarrow R$ satisfies universally going-down if and only if the canonical map $S[x_1, x_2, \ldots, x_n] \hookrightarrow R[x_1, x_2, ..., x_n]$ satisfies going-down for each n.

Note that if G acts on R, then this action naturally extends to $R[x_1, x_2, \ldots, x_n]$.

Theorem

If G is locally finite on R, then $R^G \hookrightarrow R$ is universally going-down.

We did not do this at one time.

Steps in Proof

We first proved this when G is locally finite and R Noetherian (what we really needed was that there are only finitely many primes minimal over an arbitrary ideal).

Continuation

Then we realized, using a criteria of Kaplansky, that the obstruction to going-down was a finite data set. Basically, if the inclusion R^G into R does not satisfy universally going-down, then there exists finitely many elements $a_1, a_2, \ldots, a_n \in R$ that screw things up. Take these elements and their orbits (which still leave you with a finite set) and take the ring generated by $\mathbb{Z}$ and these finite number of elements. This ring, call it T, is Noetherian, and G still acts on this ring. By construction $T^G \hookrightarrow T$ does not satisfies universally going-down. But by earlier result it does - a contradiction.

For a while we did not have an example to show that if we dropped the locally finite assumption, then $R^G \hookrightarrow R$ does not satisfy GD. But finally we did come up with an example using a semigroup ring.

Outline of the Construction

- We construct an abelian semigroup S (under $+$) via generators and relations.
- We define an automorphism group G acting on S.
- We let $R = K[S] = K[x^s : s \in S]$, where K is a field. The action of G extends in a natural fashion to R.
- We show that $R^G = K[S^G]$
- We construct the appropriate diagram and show that it cannot be completed.

The Semigroup

Let M denote the free abelian monoid on the symbols $\{A, C_i \mid i \in \mathbb{Z}\}$, written additively.

We define a congruence relation on this monoid, as follows. Let $nA + C_{i_1} + C_{i_2} + \cdots + C_{i_k}$ and $mA + C_{j_1} + \cdots + C_{j_t}$ be arbitrary elements of M, where t is a nonnegative integer, the C_k are not necessarily distinct, and n and m are nonnegative integers. Then we declare that

$$nA + C_{i_1} + C_{i_2} + \cdots + C_{i_k} \equiv mA + C_{j_1} + \cdots + C_{j_t}$$

if either $nA + C_{i_1} + C_{i_2} + \cdots + C_{i_k} = mA + C_{j_1} + \cdots + C_{j_t}$ or $[n = m \neq 0$ and $k = t]$. This is a congruence relation on M (that is, an equivalence relation on M that is compatible with the operation of addition on M). Let S denote the factor semigroup $M/\equiv$.

The Automorphism Group of S

First we define G on M. Let $g : M \to M$ be given by $g(A) = A$ and $g(C_i) = C_{i+1}$ and then extending linearly. It is clear that if $x, y \in M$ satisfy $x \equiv y$, then $g(x) \equiv g(y)$. Hence, g induces an automorphism of S, also denoted g. Then g has infinite order and $G = \langle g \rangle$ acts on S.

The Example

With $R = K[S]$, we have that $K[S^G] = R^G \subset R$ does not satisfy going-down (much less universally going-down).

The End

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