1. (2.3, #2) Let $K \subset L$ be fields. Let T be a ring having quotient field T and let $R = T \cap K$. Prove that if T is integrally closed, then R is integrally closed.

   Proof: Note that the quotient field of R, call it H, is contained in K (it may not equal K). Let $q \in H$ be integral over R. Thus q satisfies a monic polynomial with coefficients in R. Since $R \subset T$, q is integral over T. But T is integrally closed, so $q \in T$. Since it is in K we conclude that $q \in R$. Thus R is integrally closed.

2. (2.3, #3) Let R, T, K and L be as in the previous problem. If R is a valuation domain with maximal ideal M, prove that M survives in T.

   Proof: Assume false and say that $MT = T$. Then there exists nonzero elements $m_1, m_2, \ldots, m_s \in M$ and $t_1, t_2, \ldots, t_s \in T$ such that

   $$m_1t_1 + m_2t_2 + \ldots + m_st_s = 1.$$

   However, since R is a valuation domain, one of the m_i's divides all the others. Without loss of generality, assume that $m_1|m_j$ for $j = 2, 3, \ldots, s$. Thus m_1 can be factored out of the above equation (in T). From which we obtain

   $$m_1(t_1' + t_2' + \ldots + t_s') = 1.$$

   In particular m_1 has an inverse in T. But $m_1 \in R = T \cap K$. Since T contains both m_1 and its inverse, and since K is a field that contains m_1, K also contains m_1^{-1}. Hence $m_1^{-1} \in R$, which contradicts the fact that $m_1 \in M$.

3. (2.3 #8) Let R be an integral domain, Q a prime ideal of $R[x]$ that contracts to 0. Prove that $R[x]_Q$ is a DVR.

   Proof:

   Since Q contracts to 0, all the non-zero elements of R are in $R[x] \setminus Q$. So let $S = R \setminus \{0\}$. Then S is a multiplicatively closed subset of $R[x]$ which is contained in $R[x] \setminus Q$. Thus $R[x]_Q = (R[x]_S)_{Q'}$, where Q' is the image of Q in $R[x]_S$. But $R[x]_S = K[x]$ is a PID. Hence either $Q' = 0$, in which case $R[x]_Q = K_{Q'}$ is a field, or $K[x]_{Q'}$ is a local PID, which is a DVR.

4. (2.3 #11) Prove that if every prime ideal of the domain R is invertible, then R is Dedekind.

   Proof: Since invertible ideals are finitely generated, we can conclude that all the prime ideals are finitely generated. By an earlier Theorem, we can conclude that all ideals are finitely generated, i.e., R is Noetherian. Let I be an arbitrary ideal, we have to show that I is invertible, namely that $II^{-1} = R$.

   First we will show that I can be written as a product of prime ideals. We know that $I \subset P$ where P is a prime ideal. We can assume that $I \neq P$, otherwise we are done. We can write $I = (IP^{-1})P$. If IP^{-1} is prime, then we have proved the claim. Also note that $I \subset IP^{-1}$, since $R \subset P^{-1}$. Moreover the containment is strict by what we did in class. So we can then rewrite IP^{-1} as a product of a prime ideal and an ideal that is strictly larger than itself. Since R is Noetherian, this process must stop. Thus the claim is proved and $I = P_1 P_2 \cdots P_n$, each P_i prime.

   Next we show that if A and B are ideals of R, then $A^{-1}B^{-1} \subseteq (AB)^{-1}$. An arbitrary element of $A^{-1}B^{-1}$ has the form $\sum x_i y_i$, $x_i \in A^{-1}$ and $y_i \in B^{-1}$. So if $x \in A^{-1}$ and $y \in B^{-1}$, it suffices to show that $xy \in (AB)^{-1}$. Then $ABxy = (Ax)(Ay) \subset RR = R$. So it follows that $A^{-1}B^{-1} \subseteq (AB)^{-1}$. Now if $I = P_1 P_2 \cdots P_n$, then $I^{-1} \supseteq (P_1)^{-1}(P_2)^{-1} \cdots (P_n)^{-1}$ by what we just said. Hence $II^{-1} = P_1 \cdots P_n I^{-1} \supseteq [P_1 \cdots P_n](P_1)^{-1}(P_2)^{-1} \cdots (P_n)^{-1} = P_1(P_1)^{-1} \cdots P_n(P_n)^{-1} = R$. Thus $II^{-1} = R$, so I is invertible and we are done.