1. (1.6, #6) Let R be an integrally closed domain with integral closure T and S a multiplicatively closed set in R. Prove that the integral closure of R_S is T_S.

   Proof: Let R, T and S be as given. Let $t/s \in T_S$, (so $t \in T$ and $s \in S$). Then there exists $a_i \in R$ such that

   $$t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 = 0$$

   . Multiply the above by $\frac{1}{s^n}$ which gives

   $$\frac{1}{s^n}t^n + \frac{a_{n-1}}{s}\left(\frac{t}{s}\right)^{n-1} + \cdots + \frac{a_1}{s^{n-1}}\left(\frac{t}{s}\right) + \frac{a_0}{s^n} = 0$$

   Clearly each $\frac{a_i}{s^{n-i}} \in R_S$. Thus t/s is integral over R_S. Since t/s was arbitrary, we have that T_S is integral over R_S. Moreover by Theorem 51 T_S is integrally closed. hence it is the integral closure of R_S.

2. (1.6 #27) Let R be an integral domain with quotient field K. Suppose that every ring between R and K is integrally closed. Prove that R is Prüfer.

   Proof: To show that R is Prüfer it suffices to show that R_M is a valuation ring for each maximal ideal M. Also note that every overring of R_M is also an overring of R, and so integrally closed. Hence after localizing we may assume without loss of generality that R is also local. Let $u \in K$. We must show that either u or u^{-1} is in R. By assumption $R[u^2]$ is integrally closed and clearly u is integral over $R[u^2]$. Thus $u \in R[u^2]$. Hence we have that u is a polynomial over R in u^2, that is

   $$u = a_n(u^2)^n + a_{n-1}(u^2)^{n-1} + \cdots + a_0$$

   where $a_i \in R$. On the RHS non of the exponents of u is a 1. Thus by bringing over u to the other side, we see that u satisfies a polynomial over R with one coefficient equal to -1, which is clearly a unit of R. Thus we may apply Theorem 67 which states that either u or u^{-1} is in R. Since u was an arbitrary element of K, we can conclude that R is a valuation ring, which finishes the proof.

3. (1.6 #35) Let $R \subseteq T$ be domains with T algebraic over R and R integrally closed in T. Prove that T is contained in Frac(R).

   Proof: Let R and T be as given. Let $u \in T$, we have to show that $u = a/b$ or equivalently $bu = a$ for some $a, b \in R$. By assumption u is the root of some polynomial over R (just not monic). Say

$$a_n u^n + a_{n-1} u^{n-1} + \cdots + a_1 u + a_0 = 0$$

   where $a_i \in R$. Multiply this equation by a_n^{n-1} to get

$$(a_n u)^n + a_{n-1}(a_n u)^{n-1} + a_{n-2} a_n (a_n u)^{n-2} + \ldots + a_1 a_n^{n-2}(a_n u) + a_0 a_n^{n-1} = 0.$$

   Hence $a_n u \in T$ is in fact integral over R. But R is integrally closed in T. Thus $a_n u = b \in R$, which proves the result.

4. (Other hand-in) Let $R \subset T$ where T is integral over R. Show

   (a) If $u \in R$ is a unit of T, then u is a unit of R.

   Proof: Suppose that u is not a unit of R. Then u is contained in a maximal ideal P of R. Since T is integral over R, we know that LO is satisfied. Thus there is a prime ideal Q of T such that $Q \cap R = P$. Hence $u \in Q$. But this is impossible since u, as a unit of T, can not be in any ideal of T.

   (b) The Jacobson radical of R is the contraction of the Jacobson radical of T.

   Proof: Let M be any maximal ideal of R. By LO there exists a prime ideal Q of T such that $Q \cap R = M$. We also know by a Theorem in class that since M is maximal in R, Q must be maximal in T. Thus every maximal ideal of R is the contraction of a maximal ideal of T. By the same Theorem, every maximal ideal of T contracts to a maximal ideal of R. Hence it is clear that the Jacobson radical of R is the contraction of the Jacobson radical of T.