1. (1.4, #6) Let R be a ring with no non-zero nilpotent elements, and let P be a minimal prime ideal in R. Prove that R_P is a field.

   Proof: To show that a ring is a field it suffices to show that the ring has a unique prime ideal and that it has no non-zero nilpotent elements. The reason that this is true is that since it has a unique prime ideal M (which is necessarily a maximal ideal), that ideal is the radical of the ring and so it consists of nilpotent elements. But by the second assumption, we then have $M = 0$. Hence 0 is a maximal ideal.

   We know that there is a bijection between prime ideals of R_P and prime ideals of R contained in P. Thus PR_P is the unique prime ideal of R_P. Moreover I claim that R_P has no non-zero nilpotent elements. For suppose that $0 \neq a \in R_P$ is nilpotent. Then $a^n = 0$ some $n > 1$. By the definition of localization, there exists $\beta, \gamma \in R$ such that in R_P we have $a = b/c$, where b, c denote the images of β, γ respectively in R_P. Thus $b^n = 0$ and this means that back in R there exists $s \in R \setminus P$ such that $\beta^n s = 0$. Thus $(\beta s)^n = 0$ and so by our assumption $\beta s = 0$. Hence $b = 0$ (in R_P) and so $a = 0$. This proves the claim and so by the first statement it also proves the result.

2. (1.5 #1) Let Q be a prime ideal of $R[x]$ contracting to P in R. Prove that Q is a G-ideal if and only if P is a G-ideal of R and Q properly contains $PR[x]$.

   Proof: ($\Rightarrow$) Suppose that Q is a G-ideal. Then there exists a maximal ideal M of $R[x][y] = R[x, y]$ that contracts to Q (Theorem 27). Thus $M \cap R = P$. By the generalization of Theorem 27 that we showed in an earlier homework, P is a G-ideal. Clearly $PR[x] \subseteq Q$. If they are equal, then $R/PR[x] \cong (R/P)[x]$ is a G-domain. But this contradicts Theorem 21 which states that if you adjoin an indeterminate to any ring, what you get is NOT a G-domain.

   ($\Leftarrow$) Let P be a G-ideal of R and assume that Q properly contains $PR[x]$. To show that Q is a G-ideal, it suffices to show that there exists a $u \in R[x] \setminus Q$ such that $u \in Q'$ for every prime ideal Q' of $R[x]$ that properly contains Q (Theorem 19). Since P is a G-ideal, there exists $u \in R \setminus P$ such that $u \in P'$ for prime ideal P' of R that properly contains P. We have the chain $PR[x] \subset Q \subset Q'$ of three distinct prime ideals of $R[x]$. Notice that Q' cannot contract to P since there cannot be a chain of three primes of $R[x]$ that contract to the same prime of R (Theorem 37). Hence $Q' \cap R$ strictly contains P and so $u \in Q' \cap R \subset Q'$. Thus u is in every prime ideal of $R[x]$ that strictly contains Q.

3. (1) Let I be a decomposable ideal of R and let P be a maximal element of the set $\{(I : x)\}$ for $x \notin I$. Show that P is prime.

   Proof: Let $P = (I : y)$ be maximal in the set above. Suppose that $ab \in P$, but $b \notin P$. We must show that $a \in P$. Since $b \notin P$, $by \notin I$. Additionally, as $Py \subseteq I$, we must have $Pby \subseteq I$. In other words $P \subseteq (I : by)$. But P is a maximal element of this set. Hence $P = (I : by)$. On the other hand by assumption $aby \in I$ ($ab \in P$). Thus $a \in (I : by) = P$.

4. (2) Let I be an ideal of R and let $S = 1 + I$. First show that S is a multiplicatively closed set, and then show that I_S is contained in the Jacobson radical of R_S ($J(R_S)$).

   Proof: It is straightforward to show that S is a multiplicatively closed set. For the second part note that I_S consists of all elements of R_S whose numerator is an element of the (image) of I. Since the image of I generates the ideal I_S, it suffices to show that this image is contained in $J(R_S)$. Recall that in any ring, an element x is in the Jacobson radical iff $1 - rx$ is a unit of the ring for all r in the ring. Back to our case. Let $i \in I$ and $r \in r$ and $s \in S$. We must show that $1 + (\frac{i}{1})(\frac{r}{s}) = \frac{s+ir}{s}$ is a unit of R_S. But $s = 1 + t$ for some $t \in I$. Hence the numerator of the element is $s + ir = 1 + t + ir$. Since $t + ir \in I$, it follows that the numerator is an element of S and whence a unit of R_S. Thus the whole fraction is a unit and we are done.