1. (1.3 #6) Let $K$ be any field. Prove that any maximal ideal of $K[x_1, \ldots, x_n]$ can be generated by $n$ elements.

Proof: We induct on $n$. Since $K[x]$ is a PID, the base case is clearly true. Next note that $K[x_1, \ldots, x_i]$ is a Hilbert ring by Theorem 31. Now let $M$ be a maximal ideal of $K[x_1, \ldots, x_n]$ and let $N = M \cap K[x_1, \ldots, x_{n-1}]$. Then $N$ is a $G$-ideal by Theorem 27. But $K[x_1, \ldots, x_{n-1}]$ is a Hilbert ring, hence $N$ is a maximal ideal. Thus by Theorem 28, $M$ can be generated by $N$ and one more element. By induction, $N$ can be generated by $n-1$ elements, so we are done.

2. (1.3 #9) Prove that the following statements are equivalent:

(a) $R$ is a Hilbert ring;
(b) Every radical ideal in $R$ is the intersection of maximal ideals;
(c) Every prime ideal in $R$ is an intersection of maximal ideals;
(d) Every $G$-ideal in $R$ is an intersection of maximal ideals.

Proof: (a) $\Rightarrow$ (b) By Theorem 26 the radical of an ideal is the intersection of $G$-ideals that contain it. Since all $G$-ideals are maximal, we are done.

(b) $\Rightarrow$ (c) This is clear.

(c) $\Rightarrow$ (d) This is clear.

(d) $\Rightarrow$ (a) Let $P \subseteq R$ be a $G$-ideal of $R$. We want to show that $P$ is a maximal ideal of $R$. By replacing $R$ with $R/P$, we may assume that $R$ is a $G$-domain, and that the 0-ideal is the intersection of maximal ideals. However, in a $G$-domain, there is a non-zero element $u$ that is in every nonzero prime ideal of the ring. Thus if the 0-ideal is not a maximal ideal, then $u$ is in every maximal ideal of $R$. Thus the intersection of the maximal ideals of $R$ contains $u \neq 0$ and hence is not the 0-ideal, a contradiction.

3. (1.3 #10) Let $M$ be a maximal ideal in $T = R[x_1, x_2, \ldots, x_n]$ such that $M \cap R = 0$. Prove that $M$ can be generated by $n+1$ elements.

Proof: First note that $R$ is an integral domain, since 0 is a prime ideal as it is the contraction of a prime ideal from the polynomial ring. We show that $R$ is a $G$-domain. Note that the canonical map $R \rightarrow R[x_1, \ldots, x_n]/M$ is injective since $M \cap R = 0$. Let $u_i$ denote the image of $x_i$ in the factor ring. Thus $R[u_1, \ldots, u_n]$ equals the field $R[x_1, \ldots, x_n]/M$. Since $R[u_1, \ldots, u_{n-1}][u_n] = R[u_1, \ldots, u_n]$, by Theorem 23 we see that $R[u_1, \ldots, u_{n-1}]$ is a $G$-domain. Similarly, since $R[u_1, \ldots, u_{n-1}] = R[u_1, \ldots, u_{n-2}][u_{n-1}]$, we have $R[u_1, \ldots, u_{n-2}]$ is a $G$-domain. Eventually, we get that $R$ is a $G$-domain.

Let $K$ be the quotient field of $R$. Thus $K = R[u^{-1}]$ for some $u \in R \subseteq T$. We note that $u \notin M$, since $M \cap R = 0$. Hence the image of $u$ is a unit in the field $T/M$. Thus there exists $g \in T$ such that $ug - 1 \in M$ or $(ug - 1) \subseteq M$. Now pass to the ring $T' := T/(ug - 1)$ (not that it matters, but this maps onto $T/M$), and note that it contains a copy of $R$, since $(ug - 1) \cap R \subseteq M \cap R = 0$. We therefore can view $R$ as a subring of $T'$. Let $v$ denote the image of $g$ in $T'$. It follows that the subring $R[v] \subseteq T'$ is a field since it is isomorphic to $R[u^{-1}] = K$, where a surjective map $K \rightarrow R[v]$ is given by $r \mapsto r$ and $u^{-1} \mapsto v$. It is injective since the domain of the function is a field. Additionally, $T' = R[v][v_1, \ldots, v_n]$, where $v_i$ denotes the image of $x_i$ in $T'$.
Let $S = K[t_1, \ldots, t_n]$. Then $S$ is a homomorphic image of $K[x_1, \ldots, x_n]$, and so any maximal ideal of $S$ is the image of a maximal ideal of the polynomial ring. Thus any maximal ideal of $S$ can also be generated by $n$ elements. Now back to our problem. Let $M'$ denote the image of $M$ in $T'$. By Problem 6, $M'$ can be generated by $n$ elements. Hence if we pull back one preimage for each element of the generating set of $M'$ to an element of $M$ and then add $ug - 1$, we have a generating set for $M$ with $n + 1$ elements. Done!

4. (1.3 # 16) Let $R \subseteq T$ be rings with $R$ a Hilbert ring and $T$ finitely generated as a ring over $R$. Prove that any maximal ideal in $T$ contracts to a maximal ideal in $R$.

**Proof:** Let $T = R[u_1, \ldots, u_n]$. Then there is a natural onto ring homomorphism $\varphi : R[x_1, \ldots, x_n] \rightarrow T$ such that the canonical embedding of $R$ into $R[x_1, \ldots, x_n]$ composed with $\varphi$ equals the given embedding of $R$ into $T$ (just map $x_i \mapsto u_i$). Let $M$ be a maximal ideal of $T$ and let $M' = \varphi^{-1}(M)$. Then $M'$ is a maximal ideal of $R[x_1, \ldots, x_n]$, since $\varphi$ is a surjection. In Problem 10 we generalized Theorem 24 to the ring $R[x_1, \ldots, x_n]$. It follows that Theorem 27 can be generalized to show that if $M'$ is a maximal ideal of $R[x_1, \ldots, x_n]$, then $M' \cap R$ is a $G$-ideal of $R$. Furthermore it is easy to see that $M' \cap R = M \cap R$. Since $R$ is a Hilbert ring, it follows that $M \cap R$ is maximal.

5. (1.3 # 17) Let $R_1 \subseteq R_2 \subseteq R_3$ be rings, with $R_3$ finitely generated as a ring over $R_1$. Let $P_3$ be an ideal of $R_3$ and let $P_1 = P_3 \cap R_1$ and $P_2 = P_3 \cap R_2$. Prove: if $P_3$ and $P_1$ are maximal is $P_2$.

**Proof:** First note that since $P_2 = P_3 \cap R_2$, the canonical map $R_2 \rightarrow R_3/P_3$ has kernel $P_2$. Thus we can view $R_2/P_2$ as being contained in $R_3/P_3$. Since $P_1 = P_3 \cap R_1 = P_2 \cap R_1$, in a similar vein we can conclude that $R_1/P_1 \subseteq R_2/P_2$. Moreover $R_3/P_3$ is still finitely generated as a ring over $R_3/P_3$ (why?). Thus relabeling, with $K = R_1/P_1$, $S = R_2/P_2$, and $F = R_3/P_3$ we have the following setup:

$$K \subseteq S \subseteq F$$

with $F$ ring finite over $K$. We want to prove that if both $K$ and $F$ are fields, so is $S$.

If $K$ and $F$ are fields, I claim that $F$ is in fact integral over $K$. To see why, first observe that there is an onto ring homomorphism $K[x_1, x_2, \ldots, x_n] \rightarrow F$. Let $M$ be the kernel of this map, so $M$ is a maximal ideal of the polynomial ring. By the first problem in this homework $M$ can be generated by $n$ elements, say $f_1, f_2, \ldots, f_n$. Moreover, if one checks the proof of that hw exercise, it is clear that each $f_i$ can be chosen as a polynomial in the variable $x_i$ over $K$.

Let $a_i$ denote the image of $x_i$ in $F$. Then $F = K[a_1, a_2, \ldots, a_n]$. Moreover each $a_i$ satisfies a polynomial over $K$ (namely $f_i$) and since $K$ is a field, we can assume that they are monic. Thus $F$ is generated over $K$ by elements that are integral over $K$. Hence $F$ is integral over $K$, proving the claim.

To finish off the proof, notice that $S$ is then integral over $K$, since it is contained in $F$. By a theorem any domain integral over a field is itself a field. Hence $S$ is a field.