1. (1.3 #6) Let K be any field. Prove that any maximal ideal of $K[x_1, \ldots, x_n]$ can be generated by n elements.

   Proof: We induct on n. Since $K[x]$ is a PID, the base case is clearly true.

   Next note that $K[x_1, \ldots, x_i]$ is a Hilbert ring by Theorem 31. Now let M be a maximal ideal of $K[x_1, \ldots, x_n]$ and let $N = M \cap K[x_1, \ldots, x_{n-1}]$. Then N is a G-ideal by Theorem 27. But $K[x_1, \ldots, x_{n-1}]$ is a Hilbert ring, hence N is a maximal ideal. Thus by Theorem 28, M can be generated by N and one more element. By induction, N can be generated by $n-1$ elements, so we are done.

2. (1.3 #9) Prove that the following statements are equivalent:

   (a) R is a Hilbert ring;

   (b) Every radical ideal in R is the intersection of maximal ideals;

   (c) Every prime ideal in R is an intersection of maximal ideals;

   (d) Every G-ideal in R is an intersection of maximal ideals.

   Proof: (a) $\Rightarrow$ (b) By Theorem 26 the radical of an ideal is the intersection of G-ideals that contain it. Since all G-ideals are maximal, we are done.

   (b) $\Rightarrow$ (c) This is clear.

   (c) $\Rightarrow$ (d) This is clear.

   (d) $\Rightarrow$ (a) Let $P \subset R$ be a G-ideal of R. We want to show that P is a maximal ideal of R. By replacing R with R/P, we may assume that R is a G-domain, and that the 0-ideal is the intersection of maximal ideals. However, in a G-domain, there is a non-zero element u that is in every nonzero prime ideal of the ring. Thus if the 0-ideal is not a maximal ideal, then u is in every maximal ideal of R. Thus the intersection of the maximal ideals of R contains $u \neq 0$ and hence is not the 0-deal, a contradiction.

3. (1.3 # 10) Let M be a maximal ideal in $T = R[x_1, x_2, \ldots, x_n]$ such that $M \cap R = 0$. Prove that M can be generated by $n+1$ elements.

   Proof: First note that R is an integral domain, since 0 is a prime ideal as it is the contraction of a prime ideal from the polynomial ring. We show that R is a G-domain. Note that the canonical map $R \longrightarrow R[x_1, \ldots, x_n]/M$ is injective since $M \cap R = 0$. Let u_i denote the image of x_i in the factor ring. Thus $R[u_1, \ldots, u_n]$ equals the field $R[x_1, \ldots, x_n]/M$. Since $R[u_1, \ldots, u_{n-1}][u_n] = R[u_1, \ldots, u_n]$, by Theorem 23 we see that $R[u_1, \ldots, u_{n-1}]$ is a G-domain. Similarly, since $R[u_1, \ldots, u_{n-1}] = R[u_1, \ldots, u_{n-2}][u_{n-1}]$, we have $R[u_1, \ldots, u_{n-2}]$ is a G-domain. Eventually, we get that R is a G-domain.

   Let K be the quotient field of R. Thus $K = R[u^{-1}]$ for some $u \in R \subset T$. We note that $u \notin M$, since $M \cap R = 0$. Hence the image of u is a unit in the field T/M. Thus there exists $g \in T$ such that $ug - 1 \in M$ or $(ug - 1) \subseteq M$. Now pass to the ring $T' := T/(ug - 1)$ (not that it matters, but this maps onto T/M), and note that it contains a copy of R, since $(ug-1) \cap R \subseteq M \cap R = 0$. We therefore can view R as a subring of T'. Let $\overline{g}$ denote the image of g in T'. It follows that the subring $R[\overline{g}] \subset T'$ is a field since it is isomorphic to $R[u^{-1}] = K$, where a surjective map $K \to R[\overline{g}]$ is given by $r \mapsto r$ and $u^{-1} \mapsto \overline{g}$. It is injective since the domain of the function is a field. Additionally, $T' = R[\overline{g}][v_1, \ldots, v_n]$, where v_i denotes the image of x_i in T'.

We now apply the following generalization of Problem 6: Let S be any ring containing a field K such that $S = K[t_1, \ldots, t_n]$. Then S is a homomorphic image of $K[x_1, \ldots, x_n]$, and so any maximal ideal of S is the image of a maximal ideal of the polynomial ring. Thus any maximal ideal of S can also be generated by n elements. Now back to our problem. Let M' denote the image of M in T'. By Problem 6, M' can be generated by n elements. Hence if we pull back one preimage for each element of the generating set of M' to an element of M and then add $ug - 1$, we have a generating set for M with $n + 1$ elements. Done!

4. (1.3 # 16) Let $R \subset T$ be rings with R a Hilbert ring and T finitely generated as a ring over R. Prove that any maximal ideal in T contracts to a maximal ideal in R.

   Proof: Let $T = R[u_1, \ldots, u_n]$. Then there is a natural onto ring homomorphism $\varphi : R[x_1, \ldots, x_n] \longrightarrow T$ such that the canonical embedding of R into $R[x_1, \ldots, x_n]$ composed with φ equals the given embedding of R into T (just map $x_i \mapsto u_i$). Let M be a maximal ideal of T and let $M' = \varphi^{-1}(M)$. Then M' is a maximal ideal of $R[x_1, \ldots, x_n]$, since φ is a surjection. In Problem 10 we generalized Theorem 24 to the ring $R[x_1, \ldots, x_n]$. It follows that Theorem 27 can be generalized to show that if M' is a maximal ideal of $R[x_1, \ldots, x_n]$, then $M' \cap R$ is a G-ideal of R. Furthermore it is easy to see that $M' \cap R = M \cap R$. Since R is a Hilbert ring, it follows that $M \cap R$ is maximal.

5. (1.3 # 17) Let $R_1 \subset R_2 \subset R_3$ be rings, with R_3 finitely generated as a ring over R_1. Let P_3 be an ideal of R_3 and let $P_1 = P_3 \cap R_1$ and $P_2 = P_3 \cap R_2$. Prove: if P_3 and P_1 are maximal is is P_2.

   Proof: First note that since $P_2 = P_3 \cap R_2$, the canonical map $R_2 \to R_3/P_3$ has kernel P_2. Thus we can view R_2/P_2 as being contained in R_3/P_3. Since $P_1 = P_3 \cap R_1 = P_2 \cap R_1$, in a similar vein we can conclude that $R_1/P_1 \subset R_2/P_2$. Moreover R_3/P_3 is still finitely generated as a ring over R_1/P_1 (why?). Thus relabeling, with $K = R_1/P_1$, $S = R_2/P_2$, and $F = R_3/P_3$ we have the following setup:

   $$K \subseteq S \subseteq F$$

   with F ring finite over K. We want to prove that if both K and F are fields, so is S.

   If K and F are fields, I claim that F is in fact integral over K. To see why, first observe that there is an onto ring homomorphism $K[x_1, x_2, ...x_n] \to F$. Let M be the kernel of this map, so M is a maximal ideal of the polynomial ring. By the first problem in this homework M can be generated by n elements, say $f_1, f_2, ...f_n$. Moreover, if one checks the proof of that hw exercise, it is clear that each f_i can be chosen as a polynomial in the variable x_i over K.

   Let a_i denote the image of x_i in F. Then $F = K[a_1, a_2, ..., a_n]$. Moreover each a_i satisfies a polynomial over K (namely f_i) and since K is a field, we can assume that they are monic. Thus F is generated over K by elements that are integral over K. Hence F is integral over K, proving the claim.

   To finish off the proof, notice that S is then integral over K, since it is contained in F. By a theorem any domain integral over a field is itself a field. Hence S is a field.