Math 724 **Solutions to HW #1** **Fall 2014**

(1) (1.1 5a) Let $P = (p)$ be a principal prime ideal and $J = \cap P^n$. If Q is a prime ideal properly contained in P, prove that $Q \subseteq J$.

Proof: The hint essentially gives this one away. Let $q \in Q$, we have to show that $q \in J$. To do that we have to show that for all $n > 0$, p^n divides q. Since $q \in P$, we know that $q = pq_1$. Since $p \notin Q$ (Q is strictly smaller than P), we conclude that $q_1 \in Q$. Hence $q_1 \in P$, so $q_1 = pq_2$. So p^2 divides q. Similarly $q_2 \in Q$, so $q_2 = pq_3$. Thus p^3 divides q, etc.

(2) (1.1 5b) Show that if p is a non-zero divisor, then $J = pJ$.

Proof: Let $j \in J$. We want to show that $j \in pJ$. In other words, show that $j = pj'$ for some $j' \in J$. Since $j \in P$ (afterall $J \subseteq P$), we can write $j = px$ some $x \in R$. For any $n > 0$, $j \in P^{n+1}$. Hence $p^{n+1}|px$. Thus for some $k \in R$, $p^{n+1}k = px$. Then we have $p(p^n k - x) = 0$. Since p is not a zero-divisor, we have $p^n k - x = 0$ or $p^n k = x$. In other words, for all $n > 0$, p^n divides x. Hence $x \in J$ (or x is our j') and we are done.

(3) (1.1)(c) (Not a hand-in) With p as above, show that J is a prime ideal.

Proof: Suppose $ab \in J$, yet neither $a \in J$ nor $b \in J$. This last statement means that some power of p does not divide a nor b. Then we can write $a = p^m a_1$ and $b = p^a b_1$ where a_1 and b_1 are not in P. But $ab \in P^{n+m+1}$. Hence p^{n+m+1} divides $p^n p^m a_1 b_1$. This means that p divides $a_1 b_1$ - a contradiction.

(4) Let (p) and (q) be nonzero principal prime ideals of a ring R. Suppose that $(p) \subseteq (q)$ and that p is a nonzero divisor. Prove that $(p) = (q)$.

Proof: Assume that the containment is strict. The hypothesis assures that $p \in (q)$ or $p = aq$ some $a \in R$. Since p is a prime element (it generates a prime ideal), we have $p \mid a$ or $p \mid q$. But we are assuming that $q \notin (p)$. Thus $a = cp$ for some $c \in R$. Hence $p = aq = cpq$. Since p is not a zero divisor, we can cancel the p on both sides to get $1 = cq$. Hence q is a unit of R, which contradicts the fact that (q) is a prime ideal (prime ideals are proper ideals).

(5) * Show that for an arbitrary ring A, $x \in J(A)$ iff $1 - xy$ is a unit of A for all $y \in A$.

Proof: If r is a non-unit of a ring, it is contained in the ideal rR. Since every ideal is contained in a maximal ideal, we have that every non-unit is contained in some maximal ideal.

If $x \in J(A)$, then $xy \in J(A)$ for every $y \in A$. Thus xy is in every maximal ideal of A. Thus $1 - xy$ is not in any maximal ideal of A (if $1 - xy$ were in say M, then also $xy \in M$, so $1 \in M$ - an impossibility). Since $1 - xy$ is not in any maximal ideal, it is not in any (proper) ideal. Hence it is a unit.

Conversely, suppose that $1 - xy$ is a unit for all $y \in A$. In addition, let M be a maximal ideal such that $x \notin M$. We will arrive at a contradiction. Since M is a maximal ideal, x is a unit modulo M (A/M is a field after all). Hence there exists $y \in A$ such that $xy = 1 \mod M$ or $1 - xy \in M$. But we assumed that $1 - xy$ is a unit and we have the contradiction.

(6) * Show that a ring is quasi-local if and only if the set of non-units of A is an ideal.

Proof: If r is a non-unit of a ring, it is contained in the ideal rRA. Since every ideal is contained in a maximal ideal, we have that every non-unit is contained in some maximal ideal.

If A has a unique maximal ideal M, then all non-units are contained in M and clearly everything in M is a non-unit. Thus the set of non-units is precisely M.

Conversely, suppose that the set of non-units is an ideal M. This ideal is maximal, because if you add anything to M, by definition you are adding a unit, so the ideal blows up to R. Furthermore, if there is another maximal ideal N, then there exists $y \in N \setminus M$. But again y must be a unit - contradiction with the fact that it is in an ideal.

(7) * Prove that the ring $\mathbb{Z}_{(3)} = \{a/b \in \mathbb{Q} : 3 \nmid b\}$ is a local ring.

Proof: We know that since $\mathbb{Z}$ is a PID, so is $R := \mathbb{Z}_{(3)}$. Hence it is Noetherian. We show that the set of non-units is an ideal of the ring. Let $a/b, c/d$ be non-units, where the fractions are reduced. This happens iff 3 divides both a and c (otherwise $b/a, d/c$ are in R). We also know that 3 does not divide either b or d. Clearly if $r \in R$, then 3 still divides the numerator of $r \cdot (a/b)$, even after reducing, since we can assume that 3 does not divide the denominator of r, and 3 is a prime number.

Finally $a/b + c/d = (ad + bc)/bd$. Clearly 3 divides $ad + bc$, yet it does not divide bd. Thus there sum is a non-unit. Hence the set of non-units is an ideal.