Homework (Fall 2014) # 3

Due Wednesday, Oct. 8

Problems with a $*$ are for Hand-in

From Kaplansky:

Sec. 1.4 - 2, 3, 6* (This proof should be short)

Sec. 1.5 - 1* (This one is a bit tricky, but short if done right)

$*$ 1. Let I be a decomposable ideal in a ring R and let P be a maximal element of the set $(I : x)$, where $x \in R$ and $x \notin I$. Show that P is a prime ideal and hence is an associated prime of I. (Some questions to ponder: Why can we not use Zorn's Lemma to show that there are always maximal elements in the set of ideals of the form $(I : x)$? Why does Noetherian come in handy at this point?)

$*$ 2. Let I be an ideal of R, and let $S = 1 + I$ (i.e., $S = \{1 + a | a \in I\}$). First show that S is a multiplicatively closed set and then show that I_S is contained in the Jacobson radical of R_S.

3. Suppose that for each prime ideal P of R, R_P has no non-zero nilpotent elements. Show that R has no non-zero nilpotent elements. If each R_P is an integral domain, is R necessarily an integral domain? (You don't have to answer this last question, just think about it.)

4. If $\sqrt{I} = I$, show that I has no embedded primes.

5. In the polynomial ring $\mathbb{Z}[x]$, let $I = (4, x)$. Show that I is M-primary, where $M = (2, x)$. Furthermore, show that I is not a power of M.

6. Let $R = K[x, y, z]$ where K is a field. Let $P_1 = (x, y), P_2 = (y, z), M = (x, y, z)$; P_1 and P_2 are prime ideals and M is a maximal ideal (you don't have to prove this). Let $I = P_1P_2$, then $I = P_1 \cap P_2 \cap M^2$ (you don't have to prove this). Show that this is a reduced primary decomposition of I.

7. For any prime ideal P of R, let $\tau_P(R)$ denote the kernel of the map $R \to R_P$. Prove that

   (i) $\tau_P(R) \subseteq P$.

   (ii) $\sqrt{\tau_P(R)} = P \Leftrightarrow P$ is a minimal prime of R.