Homework (Fall 2014) # 1
Due Soon
The hand-in problems are marked with a $*$

You may ask me questions about any of the problems, even if it's a hand-in problem. For a hand-in question you may use a previous problem without proof.

The hand-in problems are worth 6 points each.

From Kaplansky:
Sec. 1.1: 5a*, b*, c, 7*
Sec. 1.2: 3

In the following A is a commutative ring with at least two elements (0 and 1):

(1) Let $x \in A$ be nilpotent. Show that $1+x$ is a unit of A. Deduce that the sum of a nilpotent and a unit is a unit.

(2) Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal is maximal.

Definition The *Jacobson radical* of a ring A, $J(A)$, is the intersection of all the maximal ideals of A (i.e., it is the set of all elements that are contained in every maximal ideal).

(3) What is the Jacobson radical of $\mathbb{Z}$?

(4) * Show that for an arbitrary ring A, $x \in J(A)$ iff $1 - xy$ is a unit of A for all $y \in A$.

Definition The ring A is called *quasi-local* if it has a unique maximal ideal. (It is called *local* if it is also Noetherian.)

(5) * Show that a ring is quasi-local if and only if the set of non-units of A is an ideal.

(6) Show that if a ring is quasi-local, then its only idempotents are 0 and 1. (Recall, an element $e \in A$ is called idempotent if $e^2 = e$. In this case $1 - e$ is also idempotent.)

(7) * Prove that the ring $\mathbb{Z}_{(3)} = \{a/b \in \mathbb{Q} : 3 \nmid b\}$ is a local ring (do not just quote results stated in class regarding localization unless you are ready to prove them).