Final Math 724 (Fall 2014)
Due Monday, Dec. 15. - By 1pm
Each question is worth 8 points

(1) Let I be an ideal of any commutative ring R. Prove that $\sqrt{I^n} = \sqrt{I}$ for any positive integer n.

(2) Let S and T be multiplicatively closed subsets of the ring R with $S \subseteq T$. Show that there is a ring homomorphism $g : R_S \to R_T$ via $g(a/s) = a/s$. (You must show that it is well defined.)

(3) Let K be an algebraically closed field and let $R = K[x_1, \ldots, x_n]$. Let $f_1, \ldots, f_s \in R$. Then the equations $f_1 = 0, \ldots, f_s = 0$ have a simultaneous solution if and only if there do not exist $g_1, \ldots, g_s \in R$ with $\sum g_i f_f = 1$.

(4) Let R be an integral domain, with quotient field K. Suppose that R contains a prime element p such that $R[p^{-1}] = K$. Prove that R is a DVR. Hint: Explain why every element of K can be written as a fraction b/p^n with $b \in R$. What can you assume about the relation between b and p? Now let $s \in R$ be arbitrary. Write $1/s$ in this form.

(5) Let x be a regular element of a Noetherian ring, P a prime ideal minimal over (x). Prove that P has height 1.

(6) Let R be a Noetherian ring, Q a G-ideal in $R[x_, \ldots, x_n]$ and $P = Q \cap R$. Prove that $\mathrm{ht}(Q) = n+ \mathrm{ht}(P)$. (This is Section 3.2/#4.)